One-dimensional granular system with memory effects

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Abstract

We consider a hybrid compressible/incompressible system with memory effects, introduced recently by Lefebvre Lepot and Maury for the description of one-dimensional granular flows. We prove a global existence result for this system without assuming additional viscous dissipation. Our approach extends the one by Cavalletti et al. for the pressureless Euler system to the constrained granular case with memory effects. We construct Lagrangian solutions based on an explicit formula using the monotone rearrangement associated to the density. We explain how the memory effects are linked to the external constraints imposed on the flow. This result can also be extended to a heterogeneous maximal density constraint depending on time and space.

Keywords: Granular flows, pressureless gas dynamics.

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Introduction

In this paper, we consider a one-dimensional model for immersed granular flows, introduced by Lefebvre-Lepot and Maury in [16] and [17]. The model consists of a system of nonlinear partial differential equations describing the solid/liquid mixture through the evolution of the density of solid particles ρ and the Eulerian velocity field u. It takes the form

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{(1a)} \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = \rho f & \text{(1b)} \\ \partial_t \gamma + u \partial_x \gamma = -p & \text{(1c)} \\ 0 \le \rho \le 1 & \text{(1d)} \\ (1-\rho)\gamma = 0, \quad \gamma \le 0, & \text{(1e)} \end{cases}$$

where p represents the pressure, and f is an external force. Equations (1a) and (1b) express the local conservation of mass and momentum, respectively. The density is confined to values between 0 (vacuum) and 1 (for simplicity of presentation), with $\rho = 1$ representing the congested state. The pressure p plays the role of Lagrange multiplier for this pointwise constraint: through the momentum equation (1b), it acts on the fluid to ensure that condition

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(1d) remains satisfied everywhere. The amount of compression that the fluid is exposed to, but cannot accommodate because of (1d), is captured in the *adhesion potential* γ, which is linked to the pressure p through Equation (1c). It expresses a memory effect, keeping track of the history of the constraint satisfaction of the system over the course of time.

The system of Equations (1a)–(1e) therefore model two very different regimes that occur in the flow: in free zones, characterized by the condition $\rho < 1$, we have a pressureless dynamics of a compressible flow. In this regime, both p and γ vanish; see (1e) and (1c). In the congested zones, characterized by $\rho = 1$, we have the dynamics of an incompressible flow. The continuity equation (1a) implies that in congested zones the velocity must satisfy $\partial_x u = 0$ so that the action of the external force f must be balanced by the pressure p, which in turn is recorded into the adhesion potential γ. Note that in [16, 17] instead of (1c) the equation

$$\partial_t \gamma + \partial_x(\gamma u) = -p \tag{2}$$

is considered, which is *formally* equivalent to (1c) since $\partial_x u = 0$ if $\rho = 1$. On the other hand, if $\rho < 1$, then γ must vanish because of (1e). We would argue, however, that the form (1c) is more natural. In fact, differentiating this equation with respect to x, we obtain

$$\partial_t\big(\partial_x \gamma\big) + \partial_x\big(u \partial_x \gamma\big) = -\partial_x p. \tag{3}$$

Subtracting this equation from (1b), we observe that the pressure term cancels, giving

$$\partial_t(\rho u - \partial_x \gamma) + \partial_x\big((\rho u - \partial_x \gamma) u\big) = \rho f. \tag{4}$$

Hence $\partial_x \gamma$ plays the role of an additional momentum. Because of the exclusion relation (1e), the adhesion potential γ can only be different from zero where $\rho = 1$, thus $\partial_x \gamma$ is absolutely continuous with respect to ρ. In principle, it is therefore possible to define a velocity v such that $\partial_x \gamma = \rho v$. We can then rewrite (4) with $w := u - v$ in the form

$$\partial_t(\rho w) + \partial_x(\rho u w) = \rho f. \tag{5}$$

In [17], the system (1) has been supplemented with a collision law that prevents elastic shocks between congested blocks. This condition can be expressed as

$$u(t^+) = \mathrm{P}_{\mathrm{Adm}_{\rho,\gamma}}\big(u(t^-)\big), \tag{6}$$

where $u(t^\pm)$ denotes the one-sided limits of the velocity at time t, and with $\mathrm{P}_{\mathrm{Adm}_{\rho,\gamma}}$ the L^2 projection onto the set of admissible velocities, defined as the L^2 closure of

$$\left\{ v \in H^1 \colon \begin{cases} \partial_x v \geq 0 \text{ a.e. on } \{\rho = 1, \gamma = 0\} \\ \partial_x v = 0 \text{ a.e. on } \{\rho = 1, \gamma < 0\} \end{cases} \right\}.$$

The memory effects exhibited by system (1) have been brought to light by Maury [17] in the case of a single solid particle. He examined a physical system formed by a vertical wall and a spherical solid particle that is immersed in a viscous liquid. The particle evolves along the horizontal axis and is submitted to an external force and to the lubrication force exerted by the liquid. The latter becomes predominant when the particle is getting closer to the wall. At first order when the distance q between the particle and the wall goes to 0, it takes the form $F_{\mathrm{lub}} = -C\eta\,\frac{\dot q}{q}$, with η the viscosity of the liquid, $C > 0$ a constant that depends on the diameter of the particle. It prevents the contact in finite time of particle and wall.

Considering the limit of vanishing liquid viscosity $\eta = \varepsilon \to 0$, Maury proved in [17] the convergence toward a hybrid system (see system (19) below) describing the two possible states of the system: *free* when $q > 0$ and *stuck* when $q = 0$. In the limit, the system involves a new variable γ, the adhesion potential, which is the residual effect of the singular lubrication force $F^{\varepsilon}_{\text{lub}}$ for $\varepsilon \to 0$. The potential describes the stickyness of the particle: even in case of a pulling external force, it may take some time before the particle takes off from the wall.

Lefebvre-Lepot and Maury have extended this idea to a one-dimensional macroscopic system of aligned solid particles: system (1) is obtained in [16] as the formal limit of

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0 & \text{(7a)} \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho u_\varepsilon^2) - \partial_x\left(\dfrac{\varepsilon}{1-\rho_\varepsilon}\partial_x u_\varepsilon\right) = \rho_\varepsilon f. & \text{(7b)} \end{cases}$$

The lubrication force is represented at this macroscopic scale by the singular viscous term $\partial_x(\frac{\varepsilon}{1-\rho_\varepsilon}\partial_x u_\varepsilon)$, which prevents, by analogy with the single particle case, the formation of congested domains $\rho = 1$ when $\varepsilon > 0$. The rigorous proof, however, of the convergence of solutions of (7) to solutions of (1) remains an open problem. The mathematical difficulty of this singular limit relies in the lack of compactness of the non-linear term $\rho_\varepsilon u_\varepsilon^2$. This kind of singular limit has nevertheless been proved in [20] (see also [21] for a result in dimension 2) on an augmented system where an additional physical dissipation is taken into account.

In this paper, we want to study the system (1) directly, without any lubrication approximation. For that purpose, we take advantage of the link between model (1) and the model of pressureless gas dynamics in one space dimension:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{(8a)} \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0. & \text{(8b)} \end{cases}$$

We establish a global existence result for weak solutions to the following system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{(9a)} \\ \partial_t(\rho u - \partial_x\gamma) + \partial_x\big((\rho u - \partial_x\gamma)u\big) = \rho f & \text{(9b)} \\ 0 \le \rho \le 1 & \text{(9c)} \\ (1-\rho)\gamma = 0, \quad \gamma \le 0. & \text{(9d)} \end{cases}$$

Among the large literature that exists for the pressureless system (8), we are interested in the recent results of Natile and Savaré [19] and Cavalletti et al. [10] that develop a Lagrangian approach based on the representation of the density ρ by its monotone rearrangement X, which is the optimal transport between the Lebesgue measure $\mathcal{L}^1_{|[0,1]}$ and ρ; see [25].

Let us also mention that the granular system (1) can be seen as a non-trivial extension of the pressureless Euler equations under maximal density constraint

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{(10a)} \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x \pi = 0 & \text{(10b)} \\ 0 \le \rho \le 1 & \text{(10c)} \\ (1-\rho)\pi = 0, \quad \pi \ge 0. & \text{(10d)} \end{cases}$$

This system has been first introduced by Bouchut et al. [6] as a model of two-phase flows and then studied by Berthelin [4] and Wolansky [27]. The results rely on a discrete approximation generalizing the sticky particle dynamics used for the pressureless system. Recently, numerical methods based on optimal transport tools have been developed for this system; see [18, 23].

For viscous fluids, i.e., Navier-Stokes systems, a theoretical existence result can be found in [22] in the case where the maximal density constraint $\rho^*(x)$ is a given function of the space variable. Recently, Degond et al. have proved in [11] the existence of global weak solutions to the Navier-Stokes system with a time and space dependent maximal constraint $\rho^*(t,x)$ that is transported by the velocity u: it satisfies the transport equation

$$\partial_t \rho^* + u\, \partial_x \rho^* = 0. \tag{11}$$

Numerical simulations are have been studied in [11, 12] with applications to crowd dynamics. This type of heterogeneous maximal constraint may be also relevant for the dynamics of floating structures; see for instance Lannes [14].

The paper is organized as follows:

In Section 1 we briefly review the literature on the pressureless gas dynamics and introduce the mathematical tools linked to a Lagrangian description. In Section 2 we explain formally how these tools can be extended to the system (9) and give our main existence result. Section 3 is devoted to the proof of this result and Section 4 presents some numerical simulations. In the last section, we extend finally the result to the special case of time and space dependent maximal density constraint that satisfies the transport equation (11).

1 Lagrangian approach for the pressureless Euler equations

The pressureless gas dynamics equations, augmented by the assumption of adhesion dynamics, has been proposed as a simple model for the formation of large scale structures in the universe such as aggregates of galaxies. It is linked to the sticky particle system introduced by Zeldovich in [28]. The work of Bouchut [5] highlights the obstacles to proving existence of classical solutions to (8) (concentration phenomena on the density, lack of uniqueness under classical entropy conditions). Since then, several different mathematical approaches have been proposed in the literature for proving the global existence of measure solutions under suitable entropy conditions (see again [5]), among which there are approximations by the discrete sticky particles dynamics [9, 19], approximation by viscous regularization [7, 26] or, more recently, derivation by a hydrodynamic limit [13].

In particular, Natile and Savaré use [19] an interesting Lagrangian characterization of the density ρ by its monotone rearrangement X to show convergence of the discrete sticky particle system as the number N of particles goes to $+\infty$. To every probability measure $\rho \in \mathcal{P}_2(\mathbb{R})$ (i.e., with finite quadratic moment $\int_{\mathbb{R}} |x|^2\, \rho(dx) < +\infty$) there is associated a unique transport $X \in K$, the closed convex cone of non-decreasing maps in $L^2(0,1)$, such that

$$\rho_t = (X_t)_{\#} \mathcal{L}^1_{|[0,1]}. \tag{12}$$

Here $\mathcal{L}^1_{|[0,1]}$ is the one-dimensional Lebesgue measure restricted to the interval $[0,1]$ and $\#$ denotes the *push-forward* of measures, defined for all Borel maps $\zeta : \mathbb{R} \to [0,\infty]$ by

$$\int_{\mathbb{R}} \zeta(x)\, \rho_t(dx) = \int_0^1 \zeta\big(X_t(y)\big)\, dy. \tag{13}$$

If now (ρ, u) is a solution in the distributional sense of (8), then u_t can be associated to the Lagrangian velocity $U_t := \dot{X}_t$ (in the sequel all the Lagrangian variables will be denoted by capital letters and the Eulerian ones by the corresponding small letters) through

$$U_t(y) = u_t\big(X_t(y)\big). \tag{14}$$

In [19], Natile and Savaré show different characterizations of the transport X associated to an Eulerian solution of (8), in particular they prove that

$$X_t = \mathrm{P}_K\big(\bar{X} + t\bar{U}\big) \quad \text{for all } t \geq 0, \tag{15}$$

where P_K is the $L^2(0,1)$ projection onto the closed convex set K and $\bar{X}, \bar{U}$ are respectively the monotone rearrangement and the Lagrangian velocity associated to the initial data. The map $\bar{X} + t\bar{U}$ represents the free motion path, which is at the discrete level the transport corresponding to the case where the particles do not interact at all.

These arguments have been extended by Brenier et al. [8] to systems including an interaction between the discrete particles. This interaction is represented at the continuous level by a force $f(\rho)$ in the right-hand side of the momentum equation (8b).

Recently, Cavalletti et al. [10] have taken advantage of the formula (15) to construct directly global weak solutions to (8) without any discrete approximation by sticky particles. To this end, they define for all positive times t the transport X_t, associated to an initial data $(\bar{\rho}, \bar{u})$, by equation (15). The Lagrangian variables $\bar{X}$ and $\bar{U}$ are defined by

$$\bar{\rho} = (\bar{X})_{\#}\mu, \quad \bar{U} := \bar{u} \circ \bar{X} \tag{16}$$

for a more general reference measure μ in $\mathcal{P}_2(\mathbb{R})$ (for instance $\mu = \bar{\rho}$ and in this case $\bar{X} = \mathrm{id}$). As a consequence of the contraction property of the projection operator P_K, the map $t \mapsto X_t$ is Lipschitz continuous and thus differentiable for a.e. t, which allows us to define the Lagrangian velocity $U_t := \dot{X}_t$. Cavalletti et al. [10] introduce the subspace in $L^2(\mathbb{R}, \mu)$ formed by functions which are essentially constant where X_t is constant:

$$\mathcal{H}_{X_t} = L^2(\mathbb{R}, \mu)\text{-closure of } \big\{\varphi \circ X_t \colon \varphi \in \mathcal{D}(\mathbb{R})\big\}. \tag{17}$$

This space is a subset of the tangent cone to K at X_t, denoted by $\mathbb{T}_{X_t}K$, in which the Lagrangian velocity is contained. One can then show that U_t is the orthogonal projection of $\bar{U}$ onto the space $\mathcal{H}_{X_t}$:

$$U_t = \mathrm{P}_{\mathcal{H}_{X_t}}(\bar{U}). \tag{18}$$

This property ensures that there exists, for a.e. t, an Eulerian velocity $u_t \in L^2(\mathbb{R}, \rho_t)$ with the property that $U_t = u_t \circ X_t$. This is the key argument for recovering the weak formulations of the gas dynamics equations (8a)–(8b) in the Eulerian formulation.

By comparison, our granular system written under the pressureless form (9) involves an additional maximal density constraint $\rho \leq 1$, an additional variable γ linked to this maximal constraint, and an external force f. We explain in the next section how to extend the previous tools in order to deal with these additional constraints and variables.

2 Extension to granular flows, main result

Before announcing our existence result, we need to explain how to adapt the Lagrangian tools mentioned above when an external force and a maximal density constraint is given. A good way to do this is to come back to the microscopic approach, by nature Lagrangian, developed by Maury in [17] for a single sticky particle in contact with a wall.

Single particle case. Maury [17] proves by a vanishing viscosity limit (viscosity of liquid in which the particle is immersed), the existence of solutions to the hybrid system

$$\begin{cases} \dot q + \gamma = \bar u + \int_0^t f(s)\,ds & \text{(19a)} \\ q \geq 0, \quad \gamma \leq 0, \quad q\,\gamma = 0, & \text{(19b)} \end{cases}$$

which describes the two possible states of the system: *free* when $q > 0$ (that is, the particle evolves freely under the external force f), and *stuck* whenever $q = 0$. In this latter case, the adhesion potential is activated and is equal to

$$u^{\text{free}}(t) = \bar u + \int_0^t f(s)\,ds, \tag{20}$$

which is the velocity the particle would have if there was no wall on its trajectory. System (19) is in fact equivalent to the following second order system (see [15])

$$\begin{cases} \ddot q = f + \lambda & \text{(21a)} \\ \dot q(t^+) = \mathrm{P}_{C_{q,\gamma}(t)}\dot q(t^-) & \text{(21b)} \\ \mathrm{spt}\,(\lambda) \subset \{t\colon q(t) = 0\} & \text{(21c)} \\ \dot\gamma = -\lambda & \text{(21d)} \\ q \geq 0, \quad \gamma \leq 0, & \text{(21e)} \end{cases}$$

where $C_{q,\gamma}(t)$ denotes the set of admissible velocities

$$C_{q,\gamma}(t) = \begin{cases} \{0\} & \text{if} \quad \gamma(t^-) < 0 \\ \mathbb{R}^+ & \text{if} \quad \gamma(t^-) = 0,\ q(t) = 0 \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

It ensures that the particle cannot cross the wall and that it sticks to the wall as long as $\gamma < 0$. By comparison with system (1), an analogy can be made between the variables q and $1 - \rho$, between λ and $-\partial_x p$ and thus between γ_{micro} defined by (21d) and $-\partial_x \gamma_{\text{macro}}$.

Extension of the Lagrangian approach. Let $\rho_0 \in \mathcal{P}_2(\mathbb{R})$ denote the initial density. We assume that ρ_0 is absolutely continuous with respect to the Lebesgue measure and that its density (also denoted ρ_0, for simplicity) satisfies the maximal constraint

$$0 \leq \rho_0 \leq 1 \quad \text{a.e.} \tag{22}$$

As suggested by Cavalletti et al. [10] we set $\bar\rho = \rho_0$ as reference measure.

Definition 2.1. The set of square-integrable functions with respect to the measure $\bar{\rho}$ will be denoted $L^2(\mathbb{R}, \bar{\rho})$. Let $\langle \cdot, \cdot \rangle$ be the induced inner product. The space of p-integrable functions on the domain Ω for the Lebesgue measure will be denoted by $L^p(\Omega)$.

In the following, we will switch freely between absolutely continuous measures $\rho(dx)$ and their Lebesgue densities $\rho(x)\,dx$. The meaning will be clear from the context.

Set of admissible transports. To each ρ_t we associate a monotone transport map X_t through

$$\rho_t = (X_t)_{\#}\bar{\rho}. \tag{23}$$

To express the maximal density constraint $\rho_t \leq 1$ in terms of a constraint on the transport map X_t, we consider a maximally compressed density with the same total mass as $\bar{\rho}$, which is a characteristic function of some interval $\widetilde{I}$ of length one. For definiteness, we assume this interval to be centered around the center of mass of $\bar{\rho}$, but the construction is invariant under translation since constants can be absorbed into the transport map. Let $\widetilde{\rho}$ be the probability measure associated to the characteristic function of I_0. Let $\widetilde{X}$ be the unique nondecreasing transport map in $L^2(\mathbb{R}, \bar{\rho})$ (see [25] Theorem 2.5, for example) such that

$$\widetilde{\rho} = \widetilde{X}_{\#}\bar{\rho}. \tag{24}$$

The push forward formula implies that $\partial_y \widetilde{X}(y) > 0$ for $\bar{\rho}$-a.e. $y \in \mathbb{R}$ and

$$\widetilde{\rho}(x) = \frac{\bar{\rho}(\widetilde{X}^{-1}(x))}{\partial_y \widetilde{X}(\widetilde{X}^{-1}(x))} \quad \text{for a.e. } x \in I_0; \tag{25}$$

see [2] Lemma 5.5.3. In particular, we have $\bar{\rho}(y)/\partial_y \widetilde{X}(y) = 1$ for $\bar{\rho}$-a.e. $y \in \mathbb{R}$. The measure ρ_t in (23) is absolutely continuous with respect to the Lebesgue measure if and only if the approximate derivative $\partial_y X_t(y) > 0$ for $\bar{\rho}$-a.e. $y \in \mathbb{R}$. For ρ_t-a.e. $x \in \mathbb{R}$, we then have

$$\rho_t(x) = \frac{\bar{\rho}(X_t^{-1}(x))}{\partial_y X_t(X_t^{-1}(x))} = \frac{\bar{\rho}(X_t^{-1}(x))}{\partial_y \widetilde{X}(X_t^{-1}(x))} \frac{\partial_y \widetilde{X}(X_t^{-1}(x))}{\partial_y X_t(X_t^{-1}(x))} = \frac{\partial_y \widetilde{X}(X_t^{-1}(x))}{\partial_y X_t(X_t^{-1}(x))}. \tag{26}$$

In order to guarantee the maximal density constraint we are thus led to consider transport maps X_t such that $\partial_y \widetilde{X}(y) \leq \partial_y X_t(y)$ for $\bar{\rho}$-a.e. $y \in \mathbb{R}$. We therefore introduce the closed convex set of admissible transports maps in $L^2(\mathbb{R}, \bar{\rho})$ as follows: We define

$$\widetilde{K} := K + \widetilde{X}, \tag{27}$$

where K is the cone of monotone (more precisely: non-decreasing) maps of $L^2(\mathbb{R}, \bar{\rho})$. To the transport map $X_t \in \widetilde{K}$, we associate the monotone transport map

$$S_t := X_t - \widetilde{X} \in K. \tag{28}$$

Note that monotone maps are differentiable a.e., with nonnegative derivative.

Remark 2.2. Coming back to the definition of $\widetilde{\rho}$, we observe that the position of the interval $\widetilde{I}$ does not matter for the definition of $\widetilde{K}$ since the translations can be absorbed in K.

Formal description of the dynamics. In order to define for all times t an appropriate transport map X_t, we need to extend the notion of free transport $\bar{X} + t\bar{U}$ used in (15) to the case where the external force f is applied on the system. In particular, we need to extend the notion of free velocity, which for the pressureless system is simply the initial velocity $\bar{U}$. In our case, inspired by the microscopic case (20), we are naturally led to set

$$U_t^{\text{free}} := \bar{U} + \int_0^t f(s, X_s)\, ds, \tag{29}$$

integrating along the trajectories $t \mapsto X_t(y)$ starting at y. The free trajectory at time t would be then be given by the formula

$$X_t^{\text{free}} := \bar{X} + \int_0^t U_s^{\text{free}}\, ds, \tag{30}$$

and in analogy with (15), we consider

$$X_t := \mathrm{P}_{\widetilde{K}}(X_t^{\text{free}}) = \mathrm{P}_{\widetilde{K}}\left(\bar{X} + \int_0^t U_s^{\text{free}}\, ds\right); \tag{31}$$

see [8] for a similar formulation. Note carefully that equations (31)–(29) form a coupled system insofar as the free velocity U_t^{free} depends on the trajectory X_s itself. Establishing the existence and uniqueness of a solutions is non trivial and requires suitable assumptions on the external force f. We detail this point in Lemma 3.1 below. The associated velocity, formally defined as $U_t = \frac{d}{dt} X_t$, not only has to belong the tangent cone to K at $S_t = X_t - \widetilde{X}$, defined as

$$\mathbb{T}_{S_t} K := L^2(\mathbb{R}, \bar{\rho})\text{-closure of } T_{S_t} K \quad \text{where} \quad T_{S_t} K := \bigcup_{h>0} h\big(K - S_t\big), \tag{32}$$

it has to be constant on each congested block for a.e. t. That is, U_t must belong to the set

$$\mathcal{H}_{S_t} := \big\{ U \in L^2(\mathbb{R}, \bar{\rho}) \colon U \text{ is a.e. constant on maximal intervals in } \Omega_{S_t} \big\} \tag{33}$$

where Ω_{S_t} is the union of all non-trivial intervals on which S_t is constant. In analogy to the microscopic case (19a), we define an adhesion potential Γ_t; see (39).

Definition of weak solutions and main result. Here is our solution concept.

Definition 2.3. Given suitable initial data $(\bar{\rho}, \bar{u})$, a triple (ρ, u, γ) is called a weak solution of system (9) provided that

- (ρ, u, γ) satisfies

$$\rho_t \in \mathcal{P}_2(\mathbb{R}),\ u_t \in L^2(\mathbb{R}, \rho_t) \quad \text{for a.e. } t \in [0, T], \tag{34}$$

$$\gamma \in L^\infty\big([0, T]; W^{1,1}(\mathbb{R})\big); \tag{35}$$

- the density constraint (9c) and the exlusion principle (9d) hold almost everywhere;

- equations (9a) and (9b) are satisfied in the sense of distributions:

$$\int_0^T \int_{\mathbb{R}} \Big(\partial_t \xi(t,x) + u_t(x)\partial_x \xi(t,x)\Big)\rho_t(x)\,dx\,dt \tag{36}$$
$$= -\int_{\mathbb{R}} \xi(0,x)\,\bar\rho(x)\,dx \quad \text{for all } \xi \in \mathcal{C}_c^\infty\big([0,T)\times\mathbb{R}\big),$$

$$\int_0^T \int_{\mathbb{R}} \Big(\partial_t \varphi(t,x) + u_t(x)\partial_x \varphi(t,x)\Big)\big(\rho_t(x)u_t(x) - \partial_x\gamma_t(x)\big)\,dx\,dt \tag{37}$$
$$+ \int_0^T \int_{\mathbb{R}} \varphi(t,x)\,\rho_t(x) f_t(x)\,dx\,dt$$
$$= -\int_{\mathbb{R}} \varphi(0,x)\,\bar\rho(x)\bar u(x)\,dx \quad \text{for all } \varphi \in \mathcal{C}_c^\infty\big([0,T)\times\mathbb{R}\big).$$

Remark 2.4. As explained before, the velocity must be compatible with the flow configuration in the sense that it is non-decreasing on congested blocks; recall (33). We can write

$$\bar\rho(x)\bar u(x) = \bar\rho(x)u_0(x) - \partial_x\gamma_0(x),$$

where u_0 is the $L^2(\mathbb{R},\bar\rho)$-projection of $\bar u$ onto the tangent cone $\mathbb{T}_{S_0}K$ and the adhesion potential γ_0 is defined in analogy to (39).

Theorem 2.5. *Let $T>0$ and external force $f \in L^\infty\big(0,T;\mathrm{Lip}(\mathbb{R})\cap L^\infty(\mathbb{R})\big)$ be given. Suppose that $\bar\rho \in \mathcal{P}_2(\mathbb{R})$ with $\bar\rho \ll \mathcal{L}^1$ and $0 \le \bar\rho \le 1$ a.e., and that $\bar u \in L^2(\mathbb{R},\bar\rho)$. Define*

$$\bar X := \mathrm{id},\ \bar U := \bar u, \quad \textit{so that} \quad X_0 = X_0^{\textit{free}} = \mathrm{id},\ U_0^{\textit{free}} = \bar u,\ \rho_0 := \bar\rho. \tag{38}$$

There exists a curve $[0,T] \ni t \mapsto X_t \in \widetilde K$ that is differentiable for a.e. $t \in (0,T)$ and solves the coupled system of equations (29)–(31). The following quantities are well-defined:

$$U_t(y) := \dot X_t(y),\ \Gamma_t(y) := \int_{-\infty}^y \Big(U_t(z) - U_t^{\textit{free}}(z)\Big)\bar\rho(z)\,dz \tag{39}$$

for $y \in \mathbb{R}$ and a.e. $t \in (0,T)$. There exist $(u_t,\gamma_t) \in \mathcal{L}^2(\mathbb{R},\rho_t)\times W^{1,1}(\mathbb{R})$, such that

$$U_t = u_t \circ X_t,\ \Gamma_t = \gamma_t \circ X_t \quad \textit{where} \quad \rho_t := (X_t)_\#\bar\rho.$$

The triple (ρ,u,γ) is a global weak solution of system (9).

Remark 2.6. The assumption $f \in L^\infty\big(0,T;\mathrm{Lip}(\mathbb{R})\cap L^\infty(\mathbb{R})\big)$ can be relaxed to include a larger class of forces. We will stick to it here to simplify the presentation.

Notice that $X_0 = \mathrm{id}$ satisfies (26) for $t=0$, hence $X_0 \in \widetilde K$ as expected.

3 Construction of global weak solutions

Our proof consists of three steps. First we establish existence and uniqueness of X_t (defined by (31)) and U_t. We will prove that the velocity U_t is admissible in the sense that it belongs to the set $\mathcal{H}_{S_t}$ defined in (33). Introducing next the adhesion potential as in (39), we show in Subsection 3.2 that it is non-positive and supported in the congested domain. Finally, we check in Subsection 3.3 that the Eulerian variables (ρ,u,γ) associated to (X_t,U_t,Γ_t) with $t\in[0,T]$ satisfy the weak formulations (36)–(37) of system (9).

3.1 Definition of the transport and velocity

Let us begin by justifying the fact that we can define in a unique manner X_t for all times.

Lemma 3.1. *For all $t \in [0,T]$, there exists a unique solution (X_t, U_t^{free}) to* (29)–(31)*.*

Proof. Let $\mathcal{E} := \mathcal{C}([0,T], L^2(\mathbb{R},\bar\rho))$ endowed with the norm

$$\|X\|_{\mathcal{E}} = \max_{t\in[0,T]} e^{-2\sqrt{k}t} \|X_t\|_{L^2(\mathbb{R},\bar\rho)}$$

where k is the Lipschitz constant of the external force f. We define a map $\mathcal{T}$ by

$$\mathcal{T}(X)(t) := \mathrm{P}_{\widetilde{K}}\left(\bar X + t\bar U + \int_0^t \int_0^\tau f(s, X_s)\, ds\, d\tau\right), \quad t \in [0,T],$$

for all $X \in \mathcal{E}$. To prove the existence of a unique solution to (29)–(31) we will show that the map $\mathcal{T}$ is a contraction on $\mathcal{E}$. Consider $X^1, X^2 \in \mathcal{E}$ starting at $t = 0$ from $\bar X$ with velocity $\bar U$. Thanks to the contraction property of the projection map we have

$$\begin{aligned}
\|\mathcal{T}(X^1)(t) - \mathcal{T}(X^2)(t)\|_{L^2(\mathbb{R},\bar\rho)} &\le \left\| \int_0^t \int_0^\tau \left(f(s, X_s^1) - f(s, X_s^2)\right) ds\, d\tau \right\|_{L^2(\mathbb{R},\bar\rho)} \\
&\le \int_0^t \int_0^\tau \|f(s, X_s^1) - f(s, X_s^2)\|_{L^2(\mathbb{R},\bar\rho)}\, ds\, d\tau \\
&\le k \int_0^t \int_0^\tau \|X_s^1 - X_s^2\|_{L^2(\mathbb{R},\bar\rho)}\, ds\, d\tau \\
&\le k\|X^1 - X^2\|_{\mathcal{E}} \int_0^t \int_0^\tau e^{2\sqrt{k}s}\, ds\, d\tau \\
&\le \frac{1}{4} e^{2\sqrt{k}t} \|X^1 - X^2\|_{\mathcal{E}}
\end{aligned}$$

for all $t \in [0,T]$. We have therefore

$$\|\mathcal{T}(X^1) - \mathcal{T}(X^2)\|_{\mathcal{E}} \le \frac{1}{4}\|X^1 - X^2\|_{\mathcal{E}}.$$

Applying the Banach Fixed Point Theorem, we then conclude that there exists a unique map $t \mapsto X_t$ solution of (31) as well as a unique U_t^{free} for all times. □

We now recall two useful lemmas proved in [10].

Lemma 3.2 (Lemma 3.1 [10])**.** *For given $S \in L^2(\mathbb{R},\bar\rho)$ monotone, define $\Pi_S := (\mathrm{id}, S)_\#\bar\rho$. Then there exists a Borel set N_S such that $\bar\rho(N_S) = 0$ and*

$$(y, S(y)) \in \operatorname{spt}\Pi_S \quad \text{for all } y \in \mathbb{R}\setminus N_S.$$

Lemma 3.3 (Steps 1 and 2 of Lemma 3.7 [10])**.** *Assume that N_S is the $\bar\rho$-null set associated to a monotone map $S \in L^2(\mathbb{R},\bar\rho)$, as introduced in the previous lemma. Let*

$$L^z := \{y \in \mathbb{R}\setminus N_S \colon S(y) = z\},$$
$$\mathcal{O} := \{z \in \mathbb{R} \colon L^z \text{ has more than one element}\}.$$

The set $\mathcal{O}$ is at most countable and S is injective on $\mathbb{R}\setminus\bigcup_{z\in\mathcal{O}} L^z$.

Remark 3.4. Lemma 3.2 shows that the support of the transport plan Π_S induced by a map S is supported on the graph of S (which is a subset of the product space $\mathbb{R}\times\mathbb{R}$) up to a negligible set. This technical fact will be needed in the proof of Proposition 3.6 below.

Remark 3.5. To understand Lemma 3.3, recall that we are considering maps X_t in the cone $\widetilde{K} := K + \widetilde{X}$, where K denotes the cone of non-decreasing maps of $\mathcal{L}^2(\mathbb{R},\bar\rho)$ and $\widetilde{X}$ is a fixed monotone map. Because of (26), the density $\rho_t := (X_t)_{\#}\bar\rho$ satisfies the constraint

$$\rho_t(x) = 1 \quad\Longleftrightarrow\quad \partial_y\widetilde{X}_t(y) = \partial_y X_t(y)$$

with $x = X_t(y)$. This is equivalent to the condition $\partial_y S_t(y) = 0$ where $S_t := X_t - \widetilde{X}$. Recall that S_t is non-decreasing. We are thus led to consider points where S_t is constant on some open neighborhood. Applying Lemma 3.2 with $S \equiv S_t$ and denoting by $L_t^z, \mathcal{O}_t$ the corresponding sets defined above, we observe that these sets are precisely given by L_t^z, provided L_t^z has more than one point. By monotonicity of S_t, any such L_t^z must be an interval. There are at most countably many. For any such z we have $X_t(y) = z + \widetilde{X}(y)$ for a.e. $y \in L_t^z$, thus

$$\rho_t(x) = 1 \quad \text{for a.e. } x \in \big\{z + \widetilde{X}(y)\colon y \in L_t^z\big\}. \tag{40}$$

This defines one congested zone. Note that $\partial_y\widetilde{X}(y) > 0$ for $\bar\rho$-a.e. $y\in\mathbb{R}$ so that $\widetilde{X}$ is strictly increasing. The congested zone defined in (40) has positive length since L_t^z contains an open interval. Consequently, there can be at most countably many congested zones. Let

$$\Omega_{S_t} := \bigcup_{z\in\mathcal{O}_t} \big\{z + \widetilde{X}(y)\colon y \in L_t^z\big\}.$$

Proposition 3.6. *The velocity $U_t := \frac{d}{dt}X_t$ exists and belongs to $\mathcal{H}_{S_t}$ for a.e. $t\in(0,T)$.*

Proof. Due to the contraction property of the projection, we have

$$\begin{aligned}\|X_{t+h} - X_t\|_{L^2(\mathbb{R},\bar\rho)} &\le \left\| \int_0^t U_s^{\text{free}}\, ds \right\|_{L^2(\mathbb{R},\bar\rho)} \\ &\le h\|\bar U\|_{L^2(\mathbb{R},\bar\rho)} + \left\| \int_t^{t+h} \left(\int_0^s f(\tau, X_\tau)\, d\tau\right) ds \right\|_{L^2(\mathbb{R},\bar\rho)}\end{aligned}$$

and since $f\in L^\infty\big(0,T;L^\infty(\mathbb{R})\big)$ we deduce that

$$\|X_{t+h} - X_t\|_{L^2(\mathbb{R},\bar\rho)} \le |h|\Big(\|\bar U\|_{L^2(\mathbb{R},\bar\rho)} + C\big(\|f\|_{L^\infty}\big)\Big). \tag{41}$$

This proves that $t\mapsto X_t$ is Lipschitz continuous. Its time-derivative exists strongly and

$$U_t = \lim_{h\to 0^+} \frac{X_{t+h} - X_t}{h} = -\lim_{h\to 0^+}\frac{X_{t-h} - X_t}{h}$$

for a.e. $t\in(0,T)$. We deduce that

$$U_t \in \mathbb{T}_{S_t}K \cap \big(-\mathbb{T}_{S_t}K\big).$$

from the definition (32) of tangent cone, now $U_t\in\mathbb{T}_{S_t}K$ implies that there exist two sequences (W_t^k), (λ^k) with $W_t^k\in K$ and $\lambda^k > 0$, such that

$$U_t^k = W_t^k - \lambda^k S_t \quad \text{converges strongly to } U_t \text{ in } L^2(\mathbb{R},\bar\rho).$$

We can then extract a subsequence, still denoted (U_t^k), that converges a.e. towards U_t. For every k we denote by N^k the $\bar\rho$-null set associated to W_t^k, as introduced in Lemma 3.2. There exists a $B \subset \mathbb{R}$ with $\bar\rho(B) = 0$, such that $\bigcup_k N^k \subset B$ and

$$U_t^k(y) \longrightarrow U_t(y) \quad \text{for all } y \in \mathbb{R} \setminus B, \text{ as } k \to \infty.$$

For all $x \in \mathcal{O}_t$ and $y_1, y_2 \in L_t^x \setminus B$ (see Remark 3.5 for notation), we have

$$\begin{aligned}(y_1 - y_2)\big(U_t^k(y_1) - U_t^k(y_2)\big) &= (y_1 - y_2)\big(W_t^k(y_1) - \lambda^k S_t(y_1) - W_t^k(y_2) + \lambda^k S_t(y_2)\big) \\ &= (y_1 - y_2)\big(W_t^k(y_1) - W_t^k(y_2)\big) \\ &\geq 0,\end{aligned}$$

by monotonicity of W_t^k, and thus by passing to the limit $k \to +\infty$

$$(y_1 - y_2)\big(U_t(y_1) - U_t(y_2)\big) \geq 0.$$

Using now the fact that $U_t \in \big(-\mathbb{T}_{S_t} K\big)$, we obtain in the same way

$$(y_1 - y_2)\big(U_t(y_1) - U_t(y_2)\big) \leq 0.$$

Thus

$$(y_1 - y_2)\big(U_t(y_1) - U_t(y_2)\big) = 0 \quad \text{for all } x \in \mathcal{O},\ y_1,\, y_2 \in L_t^x \setminus B, \tag{42}$$

which implies that U_t belongs to $\mathcal{H}_{S_t}$. □

Proposition 3.7. *There exists a velocity $u_t \in L^2(\mathbb{R}, \rho_t)$ such that*

$$U_t(y) = u_t\big(X_t(y)\big) \quad \textit{for } \bar\rho\textit{-a.e. } y \in \mathbb{R}, \textit{ where} \quad \rho_t := (X_t)_\# \bar\rho. \tag{43}$$

Proof. Since X_t belongs to $\widetilde{K}$, for all $x \in \mathbb{R}$ there exists at most one $y \in \mathbb{R} \setminus N_{X_t}$ (where N_{X_t} is some null set associated to X_t) with $X_t(y) = x$. We can then set

$$u_t(x) := \begin{cases} U_t(y) & \text{if such } y \text{ exists} \\ 0 & \text{otherwise.} \end{cases} \tag{44}$$

Then $U_t(y) = u_t(X_t(y))$ for a.e. $y \in \mathbb{R} \setminus N_{X_t}$ and $\|u_t\|_{L^2(\mathbb{R},\rho_t)} = \|U_t\|_{L^2(\mathbb{R},\bar\rho)}$. □

Lemma 3.8. *The space $\mathcal{H}_{S_t}$ defined in* (33) *is characterized as*

$$\mathcal{H}_{S_t} = \big\{W \in L^2(\mathbb{R}, \bar\rho) \colon \textit{there exists } w \in L^2(\mathbb{R}, \eta_t) \textit{ with } W = w \circ S_t\big\}$$

where $\eta_t := (S_t)_\# \bar\rho$.

Proof. Any $W \in \mathcal{H}_{S_t}$ is essentially constant on each maximal interval of Ω_{S_t} (see Remark 3.5 for notation). For all $x \in \mathbb{R} \setminus \mathcal{O}_t$ there exists at most one $y \in L_x^t \setminus N_{S_t}$ such that $S_t(y) = x$. Let us therefore define

$$w(x) := \begin{cases} W(y) & \text{if such } y \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

For all $x \in \mathcal{O}_t$, since W is a.e. constant on L_t^x, we can pick a generic $y \in L_t^x$ and define

$$w(x) := W(y).$$

By doing so, we have constructed w such that

$$W(y) = w(S_t(y)) \quad \text{for -a.e. } x \in \mathbb{R} \setminus N_{S_t}.$$

We have then

$$\int_{\mathbb{R}} |W(y)|^2 \bar{\rho}(x)\, dy = \int_{\mathbb{R}} |w(S_t(y))|^2 \bar{\rho}(y)\, dy = \int_{\mathbb{R}} |w(x)|^2\, \eta_t(dx),$$

with η_t as defined above. $\square$

Proposition 3.9. *The space $\mathcal{H}_{S_t}$ is included in the $L^2(\mathbb{R}, \bar{\rho})$-closure of*

$$T_{S_t}K \cap \left[X_t^{free} - X_t\right]^{\perp}.$$

Proof. Due to the previous lemma, we are led to show that

$$\varphi \circ S_t \in T_{S_t}K \cap \left[X_t^{\text{free}} - X_t\right]^{\perp} \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}),$$

to get the desired result. We consider $h > \|\varphi\|_{L^\infty}$ and set

$$Z_h^{\pm} = \left(\text{id} \pm \frac{1}{h}\varphi\right) \circ S_t \in K.$$

We then have

$$\varphi \circ S_t = h(Z_h^+ - S_t),$$

which is by definition an element of the tangent cone $T_{S_t}K$; see (32). On the other hand, using the fact that X_t is the projection of X_t^{free}, we get

$$\begin{aligned} \pm\langle X_t^{\text{free}} - X_t, \varphi \circ S_t \rangle &= h\langle X_t^{\text{free}} - X_t, Z_h^{\pm} - S_t \rangle \\ &= h\langle X_t^{\text{free}} - X_t, \tilde{Z}_h^{\pm} - X_t \rangle \le 0 \end{aligned}$$

where $\tilde{Z}_h^{\pm} = Z_h^{\pm} + \widetilde{X} \in \widetilde{K}$, which proves that $\varphi \circ S_t \in \left[X_t^{\text{free}} - X_t\right]^{\perp}$. $\square$

Proposition 3.10. *The Lagrangian velocity U_t is the orthogonal projection of U_t^{free} onto $\mathcal{H}_{S_t}$.*

Proof. We already know that $U_t \in \mathcal{H}_{S_t}$. Let us therefore show that

$$\langle U_t^{\text{free}} - U_t, U_t \rangle = 0 \quad \text{and} \quad \langle U_t^{\text{free}} - U_t, W \rangle \le 0 \quad \text{for all } W \in \mathcal{H}_{S_t}. \tag{45}$$

Step 1. We that that $\langle U_t^{\text{free}} - U_t, U_t \rangle \ge 0$.
For any $t, h \in \mathbb{R}$, the quantity

$$\frac{X_{t+h} - X_t}{h}$$

is uniformly bounded. Therefore there exists a sequence (h_n) such that

$$U_t^n := \frac{X_{t+h_n} - X_t}{h_n} \longrightarrow U_t \quad \text{weakly in } L^2(\mathbb{R}, \bar{\rho}).$$

Using the fact that X_{t+h_n} is the projection of $X_{t+h_n}^{\text{free}}$ onto $\widetilde{K}$, we have the inequality

$$\langle X_{t+h_n}^{\text{free}} - X_{t+h_n}, X_t - X_{t+h_n} \rangle \leq 0,$$

which can also be rewritten as

$$\langle X_t^{\text{free}} - X_t - h_n U_t^n + \int_t^{t+h_n} U_s^{\text{free}}\, ds, -h_n U_t^n \rangle \leq 0.$$

Equivalently, by splitting the powers of h_n, we have

$$-h_n \langle X_t^{\text{free}} - X_t, U_t^n \rangle - h_n^2 \langle U_t^n - \frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}}\, ds, U_t^n \rangle \leq 0.$$

Since $h_n U_t^n = X_{t+h_n} - X_t$ and since X_t is the projection on $\widetilde{K}$ of X_t^{free}, we deduce that the first term of the left-hand side is non-negative and thus

$$-h_n^2 \langle U_t^n - \frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}}\, ds, U_t^n \rangle \leq 0.$$

As $h_n \to 0^+$ we have then

$$\frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}}\, ds \longrightarrow U_t^{\text{free}} \quad \text{strongly in } L^2(\mathbb{R}, \bar{\rho}).$$

From the weak convergence of U_t^n towards U_t, it follows that

$$\langle \frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}}\, ds, U_t^n \rangle \longrightarrow \langle U_t^{\text{free}}, U_t \rangle,$$
$$\|U_t\|_{L^2(\mathbb{R},\bar{\rho})} \leq \liminf \|U_t^n\|_{L^2(\mathbb{R},\bar{\rho})}.$$

So we finally obtain the desired inequality

$$\langle U_t^{\text{free}} - U_t, U_t \rangle \geq 0.$$

Step 2. We show that $\langle U_t^{\text{free}} - U_t, W \rangle \leq 0$ for all $W \in \mathcal{H}_{S_t}$:
Thanks to Propositions 3.6 and 3.9, there exists $h > 0$ and $Z_t \in K$ such that

$$W = h(Z_t - S_t) \quad \text{and} \quad \langle X_t^{\text{free}} - X_t, Z_t - S_t \rangle = 0.$$

We must show that

$$\langle U_t^{\text{free}} - U_t, Z_t - S_t \rangle \leq 0.$$

We consider as before the approximate velocity U_t^n and introduce $\delta_n := U_t^n - U_t$. Since X_{t+h_n} is the projection of $X_{t+h_n}^{\text{free}}$ onto $\widetilde{K}$, we have

$$\begin{aligned} 0 &\geq \langle X_{t+h_n}^{\text{free}} - X_{t+h_n}, Z_t + \widetilde{X} - X_{t+h_n} \rangle \\ &= \langle X_t^{\text{free}} - X_t + h_n \left(\frac{1}{h_n} \int_t^{t+h_n} U_s^{\text{free}}\, ds - U_t \right) - h_n \delta_n, (Z_t - S_t) - h_n U_t - h_n \delta_n \rangle. \end{aligned}$$

Rearranging the terms, we can then get

$$
\begin{aligned}
&h_n\langle \frac{1}{h_n}\int_t^{t+h_n} U_s^{\text{free}}\,ds - U_t, Z_t - S_t\,\rangle \\
&\quad\leq -\langle X_t^{\text{free}} - X_t, Z_t - S_t\,\rangle + h_n\langle X_t^{\text{free}} - X_t, U_t\,\rangle \\
&\qquad + h_n\Big(\langle X_t^{\text{free}} - X_t, \delta_n\,\rangle + \langle \delta_n, Z_t - S_t\,\rangle\Big) \\
&\qquad + h_n^2\Big(\langle \frac{1}{h_n}\int_t^{t+h_n} U_s^{\text{free}}\,ds - U_t, U_t\,\rangle + \langle \frac{1}{h_n}\int_t^{t+h_n} U_s^{\text{free}}\,ds - U_t, \delta_n\,\rangle\Big) \\
&\qquad + h_n^2\Big(\langle \delta_n, U_t\,\rangle - \|\delta_n\|^2_{L^2(\mathbb{R},\bar\rho)}\Big).
\end{aligned}
$$

By definition of Z_t and Proposition 3.9, the first line of the right-hand side vanishes. Dividing now by h_n and letting $h_n \to 0$, the remaining terms tend to 0 and we get

$$\langle U_t^{\text{free}} - U_t, Z_t - S_t\,\rangle \leq 0. \tag{46}$$

It follows that U_t is the orthogonal projection of the free velocity U_t^{free} onto $\mathcal{H}_{S_t}$. □

Remark 3.11. Since U_t is the orthogonal projection of U_t^{free} onto $\mathcal{H}_{S_t}$, we have

$$U_t(y) = \begin{cases} U_t^{\text{free}}(y) & \text{if } y \in \mathbb{R}\setminus\Omega_t \\ \dfrac{1}{\bar\rho(I)}\displaystyle\int_I U_t^{\text{free}}(z)\,\bar\rho(dz) & \text{if } y \in I \text{ with } I \in \mathcal{J}(\Omega_t), \end{cases} \tag{47}$$

where $\Omega_t := \bigcup_{z\in\mathcal{O}_t} L_t^z$ and $\mathcal{J}(\Omega_t)$ denotes the set of maximal intervals contained in Ω_t (there are at most countably many). We refer the reader to Remark 3.5 for notation.

Recovering of the mass equation. As explained in Section 2, the density

$$\rho_t := (X_t)_\#\bar\rho, \quad X_t \in \widetilde{K}, \tag{48}$$

is absolutely continuous with respect to the Lebesgue measure and satisfies the constraint

$$0 \leq \rho_t \leq 1 \quad \text{a.e.} \tag{49}$$

For all $\xi \in \mathcal{C}_c^\infty([0,T)\times\mathbb{R})$ we have by a change a variable

$$
\begin{aligned}
-\int_{\mathbb{R}} \xi(0,y)\,\bar\rho(y)\,dy &= \int_0^T \frac{d}{dt}\left(\int_{\mathbb{R}} \xi(t,X_t(y))\,\bar\rho(y)\,dy\right)\,dt \\
&= \int_0^T\int_{\mathbb{R}} \left(\partial_t\xi(t,X_t(y)) + \frac{d}{dt}X_t(y)\partial_x\xi(t,X_t(y))\right)\bar\rho(y)\,dy\,dt \\
&= \int_0^T\int_{\mathbb{R}} \Big(\partial_t\xi(t,X_t(y)) + U_t(y)\partial_x\xi(t,X_t(y))\Big)\bar\rho(y)\,dy\,dt,
\end{aligned}
$$

which gives the weak formulation of the mass equation (9a) (see Proposition 3.7):

$$-\int_{\mathbb{R}} \xi(0,x)\,\rho_0(x)\,dx = \int_0^T\int_{\mathbb{R}} (\partial_t\xi(t,x) + u_t(x)\partial_x\xi(t,x))\,\rho_t(x)\,dx\,dt. \tag{50}$$

3.2 Memory effects, definition of the adhesion potential

By analogy with the discrete model (19), we define the adhesion potential

$$\Gamma_t(y) := \int_{-\infty}^{y} \big(U_t(z) - U_t^{\mathrm{free}}(z)\big)\,\bar\rho(dz) \quad \text{for a.e. } t \in (0,T),\ y \in \mathbb{R}. \tag{51}$$

Proposition 3.12. *For a.e.* $t \in (0,T)$*,* $y \in \mathbb{R}$*, we have* $\Gamma_t(y) \le 0$ *and* $\operatorname{spt}\Gamma_t \subset \Omega_t$*.*

Proof. We use the notation introduced in Remark 3.11. For any $y \in \mathbb{R}$ let

$$\mathcal{J}_y := \big\{I \in \mathcal{J}(\Omega_t)\colon\ \sup I \le y\big\}.$$

Fix some $y \in \mathbb{R} \setminus \Omega_t$. We decompose the integral defining $\Gamma_t(y)$ and use (47) to write

$$\begin{aligned}\Gamma_t(y) &= \int_{(-\infty,y]\setminus\Omega_t} \Big(U_t(z) - U_t^{\mathrm{free}}(z)\Big)\,\bar\rho(dz) + \sum_{I\in\mathcal{J}_y} \int_I \Big(U_t(z) - U_t^{\mathrm{free}}(z)\Big)\,\bar\rho(dz)\\ &= 0 + \sum_{I\in\mathcal{J}_y} \left[\int_I \left(\frac{1}{\bar\rho(I)}\int_I U_t^{\mathrm{free}}(\tilde z)\bar\rho(d\tilde z)\right)\bar\rho(dz) - \int_I U_t^{\mathrm{free}}(z)\,\bar\rho(dz)\right] = 0.\end{aligned}$$

An integration by parts (with Γ_t continuous and $\partial_y S_t$ a measure) now yields

$$\langle\, \partial_y \Gamma_t, S_t\,\rangle = 0 \tag{52}$$

(see also [19] Lemma 3.10). Using (51) and (46), we obtain

$$\begin{aligned}\langle\, -\partial_y\Gamma_t, Z - S_t\,\rangle = \langle\, U_t^{\mathrm{free}} - U_t, Z - S_t\,\rangle \le 0&\\ \text{for all } Z \text{ in } K \text{ with } \langle X_t^{\mathrm{free}} - X_t, Z - S_t\rangle = 0.&\end{aligned}$$

Suppose that in addition $Z \in \mathcal{C}^1(\mathbb{R})$. Then

$$0 \ge \langle\, -\partial_y\Gamma_t, Z - S_t\,\rangle = \langle\, -\partial_y\Gamma_t, Z\,\rangle = \int_{\mathbb{R}} \Gamma_t(y)\partial_y Z(y)\,\bar\rho(dy).$$

From the arbitrariness of the test function Z, we obtain that $\Gamma_t \le 0$. □

As in Proposition 3.7, we can define for a.e. $t \in (0,T)$ an Eulerian adhesion potential

$$\gamma_t(x) := \Gamma_t(y) \quad \text{with} \quad x = X_t(y). \tag{53}$$

Exclusion relation. If ρ_t is a Borel family of probability measures satisfying the continuity equation in the distributional sense for a Borel velocity field u_t such that

$$\int_0^T \int_{\mathbb{R}} |u_t|\,\rho_t(dx)\,dt < +\infty,$$

then there exists a narrowly continuous curve $t \in [0,T] \mapsto \widetilde\rho_t \in \mathcal{P}(\mathbb{R})$ such that

$$\rho_t = \widetilde\rho_t \quad \text{for a.e. } t \in (0,T);$$

see [2, 25], for instance. Recall now that the transport X_t satisfies a Lipschitz property, which has allowed us to define the velocity U_t. We have that

$$X \quad \text{belongs to} \quad W^{1,\infty}(0,T;L^2(\mathbb{R},\bar\rho)) \tag{54}$$
$$U, U^{\mathrm{free}} \quad \text{belongs to} \quad L^\infty(0,T;L^2(\mathbb{R},\bar\rho)), \tag{55}$$

which implies that $\partial_y\Gamma$ is in $L^\infty(0,T;L^1(\mathbb{R}))$ since $\bar\rho$ is absolutely continuous with respect to the Lebesgue measure. The adhesion potential Γ_t is then bounded and continuous in space, hence γ_t is measurable and bounded. It can be paired with ρ_t, which is absolutely continuous with respect to the Lebesgue measure (with the pointwise bound $0 \le \rho_t \le 1$). We have

$$\begin{aligned}\int_{\mathbb{R}} \gamma_t(x)\rho_t(x)\,dx &= \int_{\mathbb{R}} \gamma_t(x)\big((X_t)_\#\bar\rho\big)(dx)\\ &= \int_{\mathbb{R}} \gamma_t(X_t(y))\bar\rho(y)\,dy = \int_{\mathbb{R}} \Gamma_t(y)\bar\rho(y)\,dy.\end{aligned}$$

On the other hand, recall that $\bar\rho(y)/\partial_y\widetilde{X}(y) = 1$ for $\bar\rho$-a.e. $y \in \mathbb{R}$, because of (25). Since Γ_t vanishes outside Ω_t, as shown in Proposition 3.12, we can write

$$\begin{aligned}\int_{\mathbb{R}} \Gamma_t(y)\bar\rho(y)\,dy &= \int_{\mathbb{R}} \Gamma_t(y)\bar\rho(y)\,dy = \int_{\Omega_t} \Gamma_t(y)\partial_y\widetilde{X}(y)\,dy\\ &= \int_{\Omega_t} \Gamma_t(y)\partial_y X_t(y)\,dy = \int_{\mathbb{R}} \Gamma_t(y)\partial_y X_t(y)\,dy = \int_{\mathbb{R}} \gamma_t(x)\,dx.\end{aligned}$$

For the third equality, we have used that $\partial_y X_t(y) = \partial_y\widetilde{X}(y)$ for a.e. $y \in \Omega_t$. Therefore

$$\int_{\mathbb{R}} \gamma_t(x)\rho_t(x)\,dx = \int_{\mathbb{R}} \gamma_t(x)\,dx$$

for a.e. $t \in (0,T)$. Since in addition $(1-\rho_t(x))\gamma_t(x) \le 0$ for a.e. $x \in \mathbb{R}$, we get

$$(1-\rho)\gamma = 0 \quad \text{a.e. on } (0,T)\times\mathbb{R}. \tag{56}$$

3.3 Recovering of the momentum equation

Similarly to the continuity equation, we want to recover the Eulerian momentum equation (9b) by passing to the Lagrangian coordinates. For all $\varphi \in \mathcal{C}_c^\infty\big([0,T)\times\mathbb{R}\big)$ we have

$$\begin{aligned}-\int_{\mathbb{R}} \varphi(0,x)\,\bar\rho(x)\bar u(x)\,dx &= -\int_{\mathbb{R}} \varphi(0,X_0(y))\,\bar\rho(y)\bar U(y)\,dy\\ &= \int_0^T\int_{\mathbb{R}} \frac{d}{dt}\varphi(t,X_t(y))\,\bar\rho(y)\bar U(y)\,dy\,dt.\end{aligned} \tag{57}$$

Recall our choice of initial data (38) (in particular, we have $X_0 = \mathrm{id}$), from which the first equality follows. We can expand the time derivative of the test function to obtain

$$\frac{d}{dt}\varphi(t,X_t(y)) = \partial_t\varphi(t,X_t(y)) + U_t(y)\partial_x\varphi(t,X_t(y)),$$

where we used that $\frac{d}{dt}X_t = U_t$, by definition. Using (51) and (29), we find

$$\partial_y \Gamma_t(y) = \bar{\rho}(y)\Big(U_t(y) - \bar{U}(y) - \int_0^t f(s, X_s(y))\,ds \Big)$$

for a.e. $y \in \mathbb{R}$. Rearranging terms, we obtain from this the identity

$$\bar{\rho}(y)\bar{U}(y) = \bar{\rho}(y)\Big(U_t(y) - \int_0^t f(s, X_s(y)) \Big) - \partial_y \Gamma_t(y),$$

which we insert into (57). Let us discuss the different terms. First, we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \frac{d}{dt}\varphi(t, X_t(y))\, \bar{\rho}(y) U_t(y)\,dy\,dt \\ &\quad = \int_0^T \int_{\mathbb{R}} \Big(\partial_t \varphi(t, X_t(y)) + U_t(y) \partial_x \varphi(t, X_t(y)) \Big)\, \bar{\rho}(y) U_t(y)\,dy\,dt \\ &\quad = \int_0^T \int_{\mathbb{R}} \Big(\partial_t \varphi(t, x) + u_t(x) \partial_x \varphi(t, x) \Big)\, \rho_t(x) u_t(x)\,dx\,dt, \end{aligned}$$

where we used Proposition 3.7. Second, by integrating by parts in time, we get

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \frac{d}{dt}\varphi(t, X_t(y))\, \bar{\rho}(y) \Big(- \int_0^t f(s, X_s(y))\,ds \Big)\,dy\,dt \\ &\quad = \int_0^T \int_{\mathbb{R}} \varphi(t, X_t(y))\, \bar{\rho}(y) f(t, X_t(y))\,dy\,dt \\ &\quad = \int_0^T \int_{\mathbb{R}} \varphi(t, x)\, \rho_t(x) f(t, x)\,dx\,dt. \end{aligned}$$

Finally, from definition (53) and the chain rule, we obtain the identity

$$\partial_y \Gamma_t(y) = \partial_y \Big(\gamma_t(X_t(y)) \Big) = \partial_x \gamma_t(X_t(y))\, \partial_y X_t(y)$$

for a.e. $y \in \mathbb{R}$. It then follows that

$$\begin{aligned} &- \int_0^T \int_{\mathbb{R}} \frac{d}{dt}\varphi(t, X_t(y))\, \partial_y \Gamma_t(y)\,dy\,dt \\ &\quad = - \int_0^T \int_{\mathbb{R}} \Big(\partial_t \varphi(t, X_t(y)) + U_t(y) \partial_x \varphi(t, X_t(y)) \Big)\, \partial_x \gamma_t(X_t(y))\, \partial_y X_t(y)\,dy\,dt \\ &\quad = - \int_0^T \int_{\mathbb{R}} \Big(\partial_t \varphi(t, x) + u_t(x) \partial_x \varphi(t, x) \Big)\, \partial_x \gamma_t(x)\,dx\,dt. \end{aligned}$$

Combining all terms, we find the momentum equation (9b), which concludes the proof.

Remark 3.13. We have uniqueness for (9) in the class of weak solutions of the form

$$\rho_t = X_t \# \bar{\rho}, \quad X_t = \mathrm{P}_{\widetilde{K}}(X_t^{\mathrm{free}}).$$

Indeed, by the contraction property of the metric projection, for solutions X_t^1, X_t^2 we have

$$\begin{aligned}\|X_t^1 - X_t^2\|_{L^2(\mathbb{R},\bar\rho)} &= \left\| \mathrm{P}_{\widetilde K}\left(\mathrm{id} + \int_0^t U_s^{\mathrm{free},1}\,ds\right) - \mathrm{P}_{\widetilde K}\left(\mathrm{id} + \int_0^t U_s^{\mathrm{free},2}\,ds\right)\right\|_{L^2(\mathbb{R},\bar\rho)} \\ &\le \left\| \int_0^t U_s^{\mathrm{free},1}\,ds - \int_0^t U_s^{\mathrm{free},2}\,ds \right\|_{L^2(\mathbb{R},\bar\rho)} \\ &\le t\|\bar U^1 - \bar U^2\|_{L^2(\mathbb{R},\bar\rho)} + k\int_0^t\int_0^s \|X_\tau^1 - X_\tau^2\|_{L^2(\mathbb{R},\bar\rho)}\,d\tau\,ds,\end{aligned}$$

with k the Lipschitz constant of f. From Gronwall's lemma (see [3] Theorem 11.4), we get

$$\|X_t^1 - X_t^2\|_{L^2(\mathbb{R},\bar\rho)} \le t\|\bar U^1 - \bar U^2\|_{L^2(\mathbb{R},\bar\rho)} + k\int_0^t \frac{s^2}{2}\exp\Big(k(t-s)\Big)\|\bar U^1 - \bar U^2\|_{L^2(\mathbb{R},\bar\rho)}\,ds, \quad (58)$$

which proves that $X_t^1 = X_t^2$ for all t if $\bar U^1 = \bar U^2$, and thus the uniqueness of the transport X_t. The velocity U_t is then uniquely defined as well since it is the orthogonal projection of

$$U_t^{\mathrm{free}} = \bar U + \int_0^t f(s, X_s)\,ds$$

onto $\mathcal{H}_{S_t}$ for a.e. t. Finally, the adhesion potential Γ_t is unique by definition (51).

Remark 3.14. The initial data is actually attained in a stronger sense than just distributionally (cf. Definition 2.3) as $t \to 0$. Let us define the L^2-Wasserstein distance

$$W_2(\rho^1,\rho^2)^2 := \min\left\{\int_{\mathbb{R}\times\mathbb{R}} |x_1 - x_2|^2\,\omega(dx_1,dx_2) \colon \omega \in \mathcal{P}(\mathbb{R}\times\mathbb{R}),\ \pi^i_{\#}\omega = \rho_i\right\}$$

where $\pi^i(x_1,x_2) = x_i$ is the projection on the ith coordinate. In the one-dimensional setting, there exists a unique optimal coupling ω: Denoting by X_i the monotone transport in $L^2(\mathbb{R},\bar\rho)$ such that $\rho_i = X_{i\#}\bar\rho$, where $\bar\rho$ is some reference measure that is absolutely continuous with respect to the Lebesgue measure, we can write

$$W_2(\rho^1,\rho^2)^2 = \int_{\mathbb{R}} |X_1(y) - X_2(y)|^2\,\bar\rho(dy);$$

see [24]). If we introduce additionally the semi-distance

$$U_2\big((\rho^1,\rho^1u^1),(\rho^2,\rho^2u^2)\big)^2 := \int_{\mathbb{R}} \big|u^1(X_1(y)) - u^2(X_2(y))\big|^2\,\bar\rho(dy),$$

then the function

$$D_2\big((\rho^1,\rho^1u^1),(\rho^2,\rho^2u^2)\big) := W_2(\rho^1,\rho^2) + U_2\big((\rho^1,\rho^1u^1),(\rho^2,\rho^2u^2)\big)$$

is a distance on the space

$$\mathcal{V}_2(\mathbb{R}) := \Big\{(\rho,\rho u) \in \mathcal{P}_2(\mathbb{R})\times\mathcal{M}(\mathbb{R}) \colon u \in L^2(\mathbb{R},\rho)\Big\}.$$

One can show that $(\mathcal{V}_2(\mathbb{R}), D_2)$ is a metric space, not necessarily complete. Convergence with respect to the distance is stronger than weak convergence of measures. We refer the reader to see [19] Proposition 2.1 and [2] Definition 5.4.3 for further information.

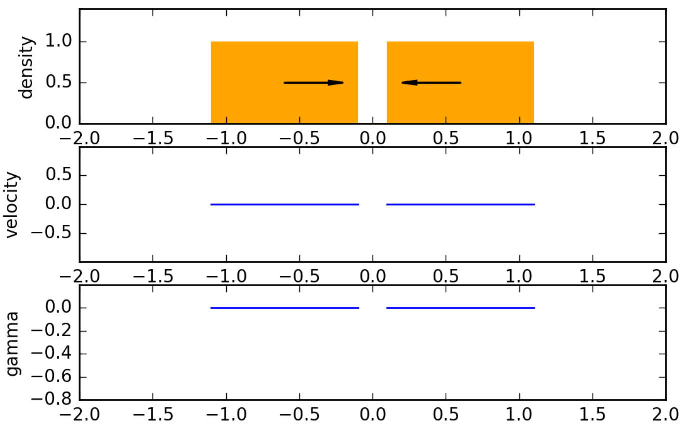

Figure 1: Initial data. The arrows represent the external force f at time 0.

The density ρ_t converges to ρ_0 for the Wasserstein distance since

$$\begin{aligned} W_2(\rho_t,\rho_0)^2 &= \int_{\mathbb{R}} \big|X_t(y)-X_0(y)\big|^2\,\bar\rho(dy)\\ &= \|X_t-\mathrm{id}\|^2_{L^2(\mathbb{R},\bar\rho)}\\ &\leq 2t\|\bar U\|^2_{L^2(\mathbb{R},\bar\rho)} + 2\left\|\int_0^t\int_0^s f(\tau,X_\tau)\,d\tau\,ds\right\|^2_{L^2(\mathbb{R},\bar\rho)} \longrightarrow 0 \end{aligned}$$

as $t\to 0$. Moreover, we can adapt the proof of [8] Theorem 3.5 to show that

$$U_t \longrightarrow \bar U \quad \text{strongly in } L^2(\mathbb{R},\bar\rho) \tag{59}$$

provided that the initial velocity $\bar U$ belongs to the tangent cone $\mathbb{T}_{S_0}K$ with $S_0 := X_0-\widetilde{X}\in K$, or even to $\mathcal{H}_{S_0}$; see (32) and (33). As follows from the proof of Proposition 3.6, this requires that the initial velocity is non-decreasing (resp. constant) on the congested zones of the initial density. If this condition is not satisfied, then the initial velocity $\bar U$ may not be attained, not even in distributional sense; see Remark 2.4. Convergence (59) translates into

$$D_2\big((\rho_t,\rho_t u_t),(\bar\rho,\bar\rho\bar u)\big) \longrightarrow 0,$$

from which it follows that $(\rho_t,\rho_t u_t)\longrightarrow(\bar\rho,\bar\rho\bar u)$ in $\mathcal{V}_2(\mathbb{R})$ as $t\to 0$.

4 Numerical simulation

To illustrate the memory effects of one-dimensional flows in (9), we consider initial data formed by two congested blocks $\mathbf{1}_{[a_1,b_1]}$ and $\mathbf{1}_{[a_2,b_2]}$ with $b_1<a_2$ at time $t=0$; see Figure 1. Initially, both the velocity and adhesion potential are equal to zero. We apply an external force f, such that the system first compresses and then decompresses in a second phase:

$$f(t,x) = \begin{cases} \alpha & \text{if} \quad x<0 \\ -\,\alpha & \text{if} \quad x\geq 0 \end{cases} \qquad \text{for} \quad t\leq t^* \tag{60}$$

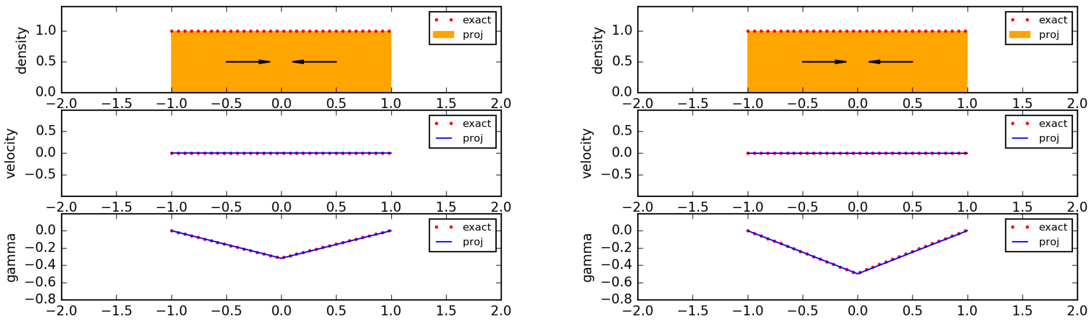

Figure 2: Solution at times $t = t_1 = 0.64$ (left) and $t = t^* = 1$ (right).

$$f(t,x) = \begin{cases} -\alpha & \text{if} \quad x < 0 \\ \alpha & \text{if} \quad x \geq 0 \end{cases} \quad \text{for} \quad t > t^*. \tag{61}$$

In our simulation, we choose $\alpha = 0.5$ and $t^* = 1$. We denote by X^{ex}, U^{ex} the exact solution of the dynamics. The process can be decomposed into four different phases:

Phase 1. The blocks move freely until time $t_1 = \sqrt{(a_2 - b_1)/\alpha}$. Then the blocks collide. We choose initial positions in such a way that the collision happens at $x = 0$:

$$\begin{cases} X_t^{\text{ex}}(y) = X_0(y) + \dfrac{\alpha t^2}{2}\Big(\mathbf{1}_{\{X_0(y)<0\}}(y) - \mathbf{1}_{\{X_0(y)>0\}}(y)\Big) \\ U_t^{\text{ex}}(y) = \alpha t\Big(\mathbf{1}_{\{X_0(y)<0\}}(y) - \mathbf{1}_{\{X_0(y)>0\}}(y)\Big) \\ \Gamma_t^{\text{ex}}(y) = 0 \end{cases} \quad \text{for } t \leq t_1.$$

Phase 2. From time t_1 on, the blocks are stuck together, but the force keeps compressing until time $t = t^*$. The velocity is 0 while the adhesion potential is activated:

$$\begin{cases} X_t^{\text{ex}}(y) = X_{t_1}^{\text{ex}}(y) \\ U_t^{\text{ex}}(y) = 0 \\ \Gamma_t^{\text{ex}}(y) = \alpha\big(X_t^{\text{ex}}(y) - (b_2 - a_2)\big)t\mathbf{1}_{\{X_0(y)>0\}}(y) \\ \qquad\qquad - \alpha\big(X_t^{\text{ex}}(y) + (b_1 - a_1)\big)t\mathbf{1}_{\{X_0(y)<0\}}(y) \end{cases} \quad \text{for } t_1 < t \leq t^*.$$

Phase 3. When we reverse the force at time $t = t^*$, the blocks remain stuck to each other until the adhesion potential comes back to 0. The velocity is zero:

$$\begin{cases} X_t^{\text{ex}}(y) = X_{t_1}^{\text{ex}}(y) \\ U_t^{\text{ex}}(y) = 0 \\ \Gamma_t^{\text{ex}}(y) = \Big[-\alpha\big(X_t^{\text{ex}}(y) - (b_1 - a_1)\big)t^* \\ \qquad\qquad + \alpha\big(X_t^{\text{ex}}(y) - (b_1 - a_1)\big)(t - t^*)\Big]\mathbf{1}_{\{X_0(y)<0\}}(y) \\ \qquad + \Big[\alpha\big(X_t^{\text{ex}}(y) - (b_2 - a_2)\big)t^* \\ \qquad\qquad - \alpha\big(X_t^{\text{ex}}(y) - (b_2 - a_2)\big)(t - t^*)\Big]\mathbf{1}_{\{X_0(y)>0\}}(y) \end{cases} \quad \text{for } t^* < t \leq t_2.$$

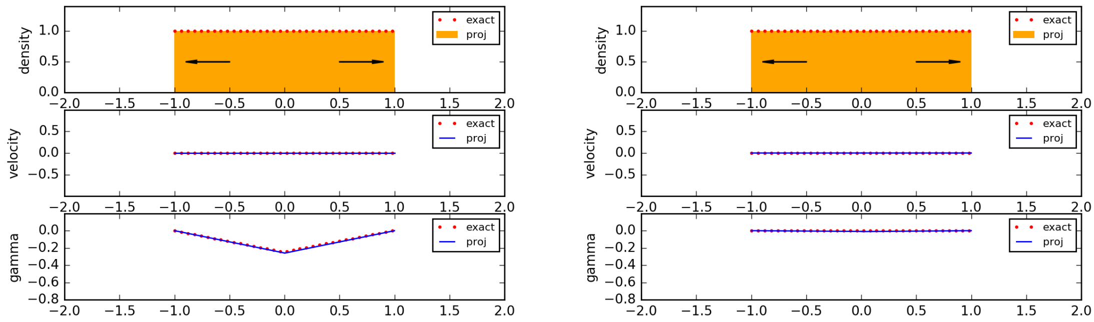

Figure 3: Solution at times $t = 1.5$ (left) and $t = t_2 = 2$ (right).

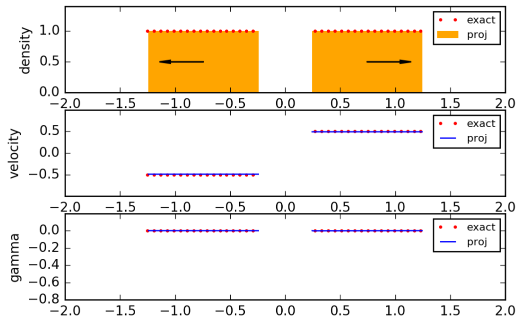

Figure 4: Solution at time $t = 3$.

Phase 4. Finally, at time $t_2 = 2t^*$ the blocks separate from each other:

$$\begin{cases} X_t^{\text{ex}}(y) = X_{t_1}^{\text{ex}}(y) + \dfrac{\alpha(t-t_2)^2}{2}\Big(- \mathbf{1}_{\{X_0(y)<0\}}(y) + \mathbf{1}_{\{X_0(y)>0\}}(y)\Big) \\ U_t^{\text{ex}}(y) = \alpha(t-t_2)\Big(- \mathbf{1}_{\{X_0(y)<0\}}(y) + \mathbf{1}_{\{X_0(y)>0\}}(y)\Big) \\ \Gamma_t^{\text{ex}}(y) = 0 \end{cases} \qquad \text{for } t \geq t_2.$$

Our numerical code follows the Lagrangian approach developed in the previous sections. To determine the transport X_{t_h} at time t_h, we minimize the objective function

$$\phi_{t_h}(X) = \left\| X_0 + \int_0^{t_h} U_s^{\text{free}}\, ds - X \right\|^2_{L^2(\mathbb{R},\rho_0}$$

under the constraint $X \in \widetilde{K}$. This step is performed by use of the Python software CVXOPT for convex optimization; see `http://cvxopt.org`. We discretize in space by considering the congested blocks consisting of two sets of equally spaced particles of equal mass $m = 1.10^{-3}$. The total number of the discrete particles in the system is here $N = 2000$. We discretize the time integral of the external force f by the left-hand rectangle method

$$\int_{t_h}^{t_h+\Delta t} f(s, X_s)\, ds \approx \Delta t\, f(t_h, X_{t_h}^{\Delta t}),$$

where $X_{t_h}^{\Delta} \in \mathbb{R}^N$ denotes the numerical solution at the discrete time $t_h := h\Delta t$ with $\Delta t > 0$. Finally, we express the constraint $X \in \widetilde{K}$ through the linear constraint

$$G\big(X_{t_h}^{\Delta t} - \widetilde{X}\big) \leq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the matrix $G \in \mathbb{R}^{N\times N}$ is given by

$$G = \begin{pmatrix} 1 & -1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & & 1 \end{pmatrix}.$$

We represent in Figures 2 and 4 our numerical solution during the four phases of the process. We observe excellent agreement between the numerical solution and the exact solution.

5 Extension to heterogeneous maximal constraint

We consider the case where in the maximal density constraint, the upper bound is replaced by a function, so that $0 \leq \rho \leq \rho^*$, where ρ^* is transported with the flow. Thus

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & (62a) \\ \partial_t(\rho u - \partial_x \gamma) + \partial_x\big((\rho u - \partial_x\gamma)u\big) = \rho f & (62b) \\ \partial_t \rho^* + u\partial_x \rho^* = 0 & (62c) \\ 0 \leq \rho \leq \rho^* & (62d) \\ (\rho^* - \rho)\gamma = 0, \quad \gamma \leq 0. & (62e) \end{cases}$$

This system has been studied by Degond et al. [11] in the Navier-Stokes framework. At initial time we prescribe, in addition to ρ_0, u_0, and γ_0, the initial constraint ρ_0^* so that $\rho_0 \leq \rho_0^*$ a.e. in $\mathbb{R}$. Combining (62a) with (62c), we observe that the ratio $r := \rho/\rho^*$ is conserved:

$$\partial_t r + \partial_x(ru) = 0. \quad (63)$$

We now consider r instead of ρ and look for a monotone transport map Y_t such that

$$r_t = (Y_t)_\# r_0.$$

We reformulate the system (62) in the form

$$\begin{cases} \partial_t r + \partial_x(ru) = 0 & (64a) \\ \partial_t(ru - \partial_x \gamma) + \partial_x\big((ru - \partial_x\gamma)u\big) = \rho f & (64b) \\ \partial_t \rho^* + u\,\partial_x \rho^* = 0 & (64c) \\ 0 \leq r \leq 1 & (64d) \\ (1 - r)\gamma = 0, \quad \gamma \leq 0. & (64e) \end{cases}$$

This transport Y_t has thus to satisfy the constraint $Y_t \in \widetilde{K}$ with $\widetilde{K} := K + \widetilde{Y}$. Here K is again the cone of monotone transport maps, and $\widetilde{Y} \in K$ is the uniquely determined transport map with $\widetilde{\rho} = \widetilde{Y}_{\#} r_0$, where $\widetilde{\rho}$ is the same as in (24). By replacing the density ρ by the ratio $r = \rho/\rho^*$ in the proof presented in the previous sections, we can define exactly in the same way as before a Lagrangian velocity $U_t = \frac{d}{dt} Y_t$ and an adhesion potential such that

$$\bar{r}(y)\bar{U}(y) = \bar{r}(y)\Big(U_t(y) - \int_0^t f(s, Y_s(y))\,ds\Big) - \partial_y \Gamma_t(y).$$

for a.e. $y \in \mathbb{R}$ and $t \in (0,T)$, with $\bar{r}, \bar{U}$ suitable initial data. One can then check that we have constructed a global weak solution to the heterogeneous system (64).

Theorem 5.1. *Let $T > 0$ and external force $f \in L^\infty\big(0,T; \mathrm{Lip}(\mathbb{R}) \cap L^\infty(\mathbb{R})\big)$ be given. Suppose that $\bar{\rho}, \rho_0^* \in \mathcal{P}_2(\mathbb{R})$ with $\bar{\rho}, \rho_0^* \ll \mathcal{L}^1$ and $\rho_0^* > 0$ a.e., and assume that*

$$0 \le \bar{\rho} \le \rho_0^* \quad \text{a.e. in } \mathbb{R}.$$

It follows that $r_0 := \rho_0^*/\bar{\rho} \le 1$. Let $\bar{u} \in L^2(\mathbb{R}, \bar{\rho})$ and define

$$\rho_0 := \bar{\rho}, \quad U_0 := \bar{u}, \quad Y_0 := \mathrm{id}.$$

There exists a curve $[0,T] \ni t \mapsto Y_t \in \widetilde{K}$ that is differentiable for a.e. $t \in (0,T)$ and solves

$$Y_t = \mathrm{P}_{\widetilde{K}}\left(Y_0 + \int_0^t U_s^{free}\,ds\right), \quad U_t^{free} = U_0 + \int_0^t f(s, Y_s)\,ds.$$

The following quantities are well-defined: $\rho_t^*(x) := \rho_0^*(Y_t^{-1}(x))$ for a.e. $x \in \mathbb{R}$, and

$$U_t(y) := \dot{Y}_t(y), \quad \Gamma_t(y) := \int_{-\infty}^y \Big(U_t(z) - U_t^{free}(z)\Big) r_0(z)\,dz$$

for $y \in \mathbb{R}$ and a.e. $t \in (0,T)$. There exist $(u_t, \gamma_t) \in \mathcal{L}^2(\mathbb{R}, r_t) \times W^{1,1}(\mathbb{R})$, such that

$$U_t = u_t \circ X_t, \quad \Gamma_t = \gamma_t \circ X_t,$$

where $r_t := (X_t)_{\#} r_0$. The tuple (r, u, γ, ρ^*) is a global weak solution of system (64).

Numerical simulation. We consider the following initial data (see Figure 5):

$$\rho_0^*(x) = 1 + 0.2 * \big(1 - \cos(2\pi(x - 0.5))\big),$$
$$\rho_0(x) = 0.8\,\rho_0^*(x)\,\mathbf{1}_{[0,1]}(x).$$

We add a compressive external force such that

$$f(x) = \begin{cases} +0.5 & \text{if } x < 0.5 \\ -0.5 & \text{if } x > 0.5, \end{cases}$$

which tends to concentrate the density at the middle of the interval $[0,1]$. We set the particle mass $m = 1.10^{-3}$, so that $N = 1000$ discrete particles are used in the simulation. We display in Figures 5 and 6 the concentration phenomenon with the appearance of a congested zone where the velocity is equal to 0 and the adhesion potential is negative.

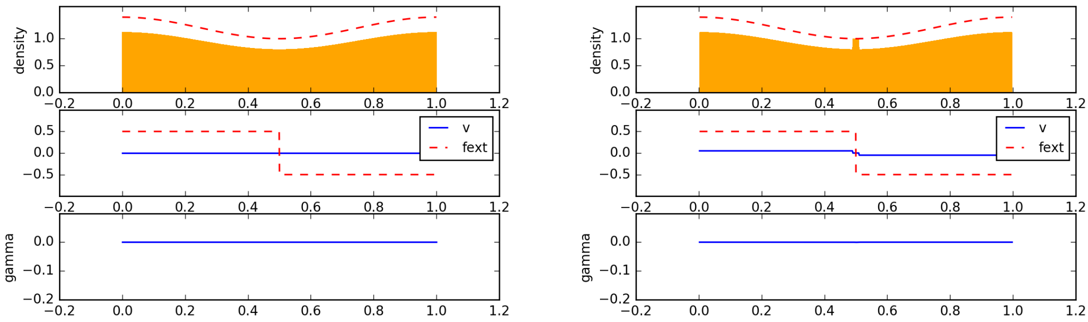

Figure 5: Variable maximal density: solution at times $t = 0$ (left) and $t = 0.1$ (right).

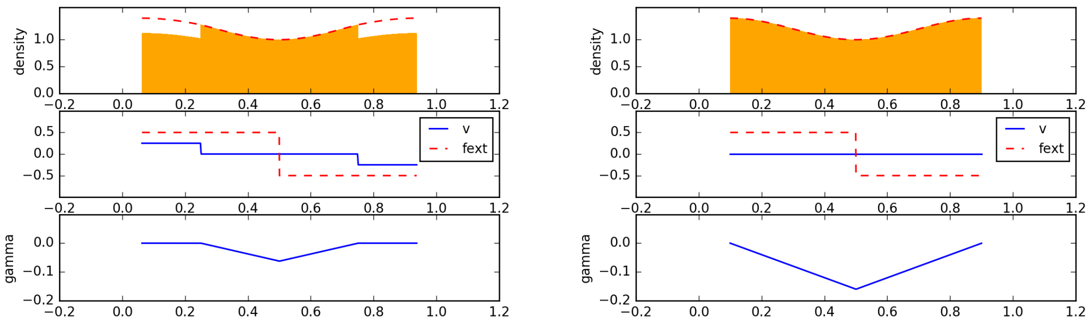

Figure 6: Variable maximal density: solution at times $t = 0.5$ (left) and $t = 0.8$ (right).

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