TOTAL OSCILLATION DIMINISHING PROPERTY FOR SCALAR CONSERVATION LAWS

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Abstract. We prove a BV estimate for scalar conservation laws that generalizes the classical Total Variation Diminishing property. In fact, for any Lipschitz continuous monotone $\Phi : \mathbb{R} \to \mathbb{R}$, we have that $|\Phi(u)|_{TV(\mathbb{R})}$ is nonincreasing in time. We call this property Total Oscillation Diminishing because it is in contradiction with the oscillations observed recently on some numerical computations based on TVD schemes. We also show that Total Variation Diminishing finite volume schemes are TOD and that the fully discrete Godunov scheme is TOD.

1. Introduction

This paper is concerned with the scalar conservation law

$$\begin{cases} \frac{\partial}{\partial t}u + \frac{\partial}{\partial x}A(u) = 0, \quad t > 0, \; x \in \mathbb{R}, \\ u(0, \cdot) = u^0 \in L^1 \cap L^\infty(\mathbb{R}). \end{cases} \tag{1}$$

In 1970, Kružkov [10] established the existence and uniqueness of entropy solutions of (1), i.e., of weak solutions $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ for which

$$\frac{\partial}{\partial t}\eta(u) + \frac{\partial}{\partial x}q(u) \leqslant 0 \quad \text{in } \mathcal{D}' \tag{2}$$

holds for all entropy-entropy flux pairs (η, q) of the family

$$\eta(u) := |u - k|, \quad q(u) := \operatorname{sgn}(u - k)\big(A(u) - A(k)\big)$$

with $k \in \mathbb{R}$. His result also generalizes to several space dimensions. The proof is based on the observation that the solution operator of (1)–(2) is an $L^1(\mathbb{R})$–contraction: for any entropy solutions u, v with initial data u^0, v^0

$$\big\|u(t) - v(t)\big\|_{L^1(\mathbb{R})} \leqslant \big\|u^0 - v^0\big\|_{L^1(\mathbb{R})}, \quad \forall t > 0.$$

This estimate implies the Total Variation Diminishing (TVD) property

$$|u(t)|_{TV(\mathbb{R})} \leqslant |u^0|_{TV(\mathbb{R})} := \big\|\tfrac{\partial}{\partial x}u^0\big\|_{M^1(\mathbb{R})}, \quad \forall t > 0. \tag{3}$$

Here $M^1(\mathbb{R})$ denotes the space of bounded Radon measures.

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Obviously, the TVD property is very useful. At the theoretical level, it provides compactness in order to prove existence. At the numerical level, it serves as a design guideline, i.e., finite volumes have been created in such a way that a discrete TVD property holds. It is usually thought that this is sufficient for stability (but note that non-entropic schemes can be TVD as well). However, two recent papers [15] and [3] show that TVD schemes can exhibit oscillations because the variation lost by the exact solution "can be used" by the scheme to create oscillations, see Figure 1. Therefore TVD is certainly not the most satisfactory property of equation (1).

In [9] the concept of *weak nonoscillation* of the entropy solution was introduced. Given some function $g \in BV(\mathbb{R})$ one considers the upper semicontinuous modification $\bar{g}$ of g (see [9]) and then counts the number of disjoint open intervals of which the level set $E_\lambda(g) := \{\bar{g} < \lambda\}$ consists, for any $\lambda \in \mathbb{R}$. Let $\mathcal{L}(E_\lambda(g))$ denote that number. A function is said to be weakly nonoscillating if it belongs to the class

$$\mathcal{W}_L := \big\{ g \in BV(\mathbb{R}) \,\big|\, \mathcal{L}\big(E_\lambda(g)\big) \leqslant L, \forall \lambda \in \mathbb{R} \big\}$$

for some $L \in \mathbb{N}$. It is shown in [9] that if the initial data $u^0 \in \mathcal{W}_L$, then the entropy solution to (1) satisfies $u(t) \in \mathcal{W}_L$ for all $t > 0$. There is also a discrete analogue of this stability property: The authors of [9] prove that certain Godunov-type schemes are weakly nonoscillatory.

In this paper, we establish a stronger stability that we state in

Theorem 1.1 (Total Oscillation Diminishing)**.** *The entropy solution to* (1) *satisfies, for all Lipschitz continuous monotone functions* $\Phi : \mathbb{R} \to \mathbb{R}$,

$$\big|\Phi\big(u(t)\big)\big|_{TV(\mathbb{R})} \leqslant \big|\Phi\big(u^0\big)\big|_{TV(\mathbb{R})}, \quad \forall t > 0. \tag{4}$$

For $\Phi(u) = u$ we recover the classical TVD property. Note, however, that we do neither assume $u^0 \in BV(\mathbb{R})$ nor that the right-hand side of (4) is finite. It might very well happen that $|\Phi(u^0)|_{TV(\mathbb{R})} < \infty$ only for certain Φ. If $|\Phi(u^0)|_{TV(\mathbb{R})} = \infty$, then the statement (4) is void, of course. Note also that Theorem 1.1 makes more precise the concept of weak nonoscillation as introduced in [9]. In fact, for $\lambda \in \mathbb{R}$ let $\Phi_\lambda(u) := \operatorname{sgn}_-(u - \lambda)$ (which is of course not Lipschitz continuous). Then

$$2\mathcal{L}\big(E_\lambda(u)\big) = \big|\Phi_\lambda(u)\big|_{TV(\mathbb{R})}.$$

One might wonder if (4) holds true for nonmonotone Φ. But this is not the case. In fact, consider a rarefaction wave for Burgers-Hopf equation, i.e., consider the flux $A(u) = \frac{1}{2}u^2$ with data $u^0 = \mathbf{1}_{(0,1)}$. If $\Phi(u) = u^2 - u$, then we obviously have $|\Phi(u^0)|_{TV(\mathbb{R})} = 0$. But u^0 evolves into a shock and a rarefaction wave which will create positive variation of $\Phi(u(t))$ for $t > 0$. Classically the TVD property (3) follows from the $L^1(\mathbb{R})$–contraction. So it would be interesting to know if

$$\big\|\Phi\big(u(t)\big) - \Phi\big(v(t)\big)\big\|_{L^1(\mathbb{R})} \tag{5}$$

is nonincreasing in time, where u, v are two entropy solutions with initial data u^0, v^0. Again such a stability result is impossible if Φ is not monotone. In fact, choose $v^0 = -u^0$ for some $u^0 \in L^1 \cap L^\infty(\mathbb{R})$. If $\Phi(u) = u^2$, then we have $\|\Phi(u^0) - \Phi(v^0)\|_{L^1(\mathbb{R})} = 0$, but usually (5) will be positive for $t > 0$. We do not know if the contraction property is true for monotone Φ.

Theorem 1.1 is a consequence of the following two propositions.

Proposition 1.2. *The entropy solution to* (1) *satisfies, for all Lipschitz continuous functions* $\Phi : \mathbb{R} \to \mathbb{R}$,

$$\big|\Phi\big(u(t)\big)\big|_{TV(\mathbb{R})} \leqslant \liminf_{\delta \to 0} \big|\Phi\big(u^0_\delta\big)\big|_{TV(\mathbb{R})}, \quad \forall t > 0. \tag{6}$$

Here $u^0_\delta \in \mathcal{S}(\mathbb{R})$ *can be any sequence with* $u^0_\delta \longrightarrow u^0$ *in* $L^1(\mathbb{R})$ *as* $\delta \to 0$.

Note that the discussion above shows that one cannot dispense with the lim inf in the initial control in (6). However, we have the following

Proposition 1.3. *Assume that the function* $\Phi : \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous and monotone. Then there exists, for all* $v \in L^1 \cap L^\infty(\mathbb{R}^n)$, *a sequence of test functions* $v_\delta \in \mathcal{S}(\mathbb{R}^n)$ *such that* $v_\delta \longrightarrow v$ *in* $L^1(\mathbb{R}^n)$ *as* $\delta \to 0$, *and*

$$\limsup_{\delta \to 0} \big|\Phi(v_\delta)\big|_{TV(\mathbb{R}^n)} \leqslant \big|\Phi(v)\big|_{TV(\mathbb{R}^n)}. \tag{7}$$

The proof of this proposition is not trivial since the usual strategy of truncating v and convolving it with some mollifier is not appropriate here. In general, this procedure would produce too much variation for $\Phi(v_\delta)$. Note also that if the conclusion of Proposition 1.3 was true for nonmonotone Φ, the same would hold for Theorem 1.1 which is not the case. Therefore monotonicity of Φ is again a necessary condition.

Proposition 1.2 follows from a Total Oscillation Diminishing (TOD) property at the parabolic level. Indeed, let $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R})$ be a solution of

$$\begin{cases} \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} A(u) = \frac{\partial^2}{\partial x^2} u, \quad t > 0,\ x \in \mathbb{R}, \\ u(0, \cdot) = u^0 \in \mathcal{S}(\mathbb{R}). \end{cases} \tag{8}$$

Note that we only consider smooth initial data. Then we have

Theorem 1.4 (Parabolic TOD property)**.** *The solution to* (8) *satisfies, for all Lipschitz continuous functions* $\Phi : \mathbb{R} \to \mathbb{R}$,

$$\big|\Phi\big(u(t)\big)\big|_{TV(\mathbb{R})} \leqslant \big|\Phi\big(u^0\big)\big|_{TV(\mathbb{R})}, \quad \forall t > 0. \tag{9}$$

We call our new property Total Oscillation Diminishing because, compared to the TVD property, it allows to focus more on specific values of the solution u and thus to avoid oscillations. In fact, consider Figure 1 which depicts the numerical solution computed by the Lax-Friedrichs scheme. (Thanks to M. Breuss for allowing us to use his pictures.) It is obtained with a timestep such that this

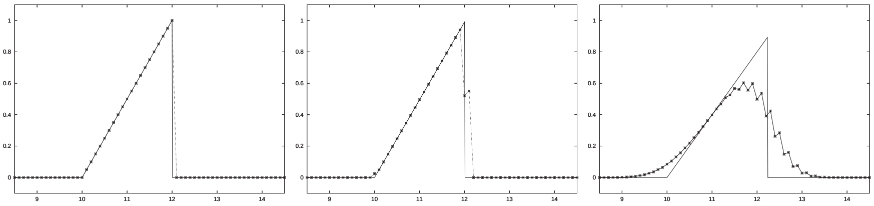

FIGURE 1. TVD schemes can create oscillations. (Pictures courtesy of M. Breuss)

scheme is TVD. But while the total variation of the solution is decreasing, new extrema are created inside the shock. Assume now that Φ is strictly increasing on some subinterval of the range of u and constant outside. Then any newly created oscillations inside that interval increase the total variation of $\Phi(u)$, in contradiction to (4).

The numerical counterpart of Theorem 1.1 is only partially covered in the present paper. We prove that semi-discrete schemes with ordered fluxes (see [12]) are Total Oscillation Diminishing, see Proposition 3.2 below. Note that ordered schemes do not necessarily enforce a discrete entropy inequality. Concerning fully discrete schemes, we prove that the classical Godunov scheme has the TOD property, see Proposition 4.1. It would be interesting to know what else schemes are Total Oscillation Diminishing.

To put the TOD property into perspective, we briefly mention some known regularity results for scalar conservation laws in one space dimension. First, Oleĭnik [11] proved for uniformly convex fluxes that entropy solutions u of (1) satisfy the one-sided Lipschitz condition

$$\left\|\left(\tfrac{\partial}{\partial x}u\right)_+\right\|_{L^\infty(\mathbb{R})} \leqslant \frac{1}{ct}, \quad \forall t>0, \tag{10}$$

where $c := \inf A''$. Note that (10) only allows for decreasing jumps, and that initial data in $L^\infty(\mathbb{R})$ is instantaneously regularized to $BV_{\mathrm{loc}}(\mathbb{R})$. Oleĭnik's result has been generalized by Hoff [8] who proves that

$$\left\|\left(\tfrac{\partial}{\partial x}a(u)\right)_+\right\|_{L^\infty(\mathbb{R})} \leqslant \frac{1}{t}, \quad \forall t>0, \tag{11}$$

where $a(u) := A'(u)$ is the characteristic speed defined by the flux. He only assumes strict convexity of the flux. Clearly, (11) implies the $BV_{\mathrm{loc}}(\mathbb{R})$–regularizing effect whenever A'' is bounded away from zero. Several attempts have been made to generalize these results to nonconvex scalar conservation laws in one space dimension. in one space dimension: Zumbrun [16] considers the fluxes $A(u) = u^k$ for $k \in \mathbb{N}$ and proves that

$$\left\|\tfrac{\partial}{\partial x}a(u)\right\|_{M^1(\mathbb{R})} \leqslant ct^{-(k-1)/k}, \quad \forall t>0.$$

Cheng [5] allows finitely many inflection points for the flux and proves that

$$\left\|\tfrac{\partial}{\partial x}a(u)\right\|_{M^1(\mathbb{R})} \leqslant \frac{c_1}{t} + c_2, \quad \forall t > 0.$$

A different result was obtained by Bouchut and James [2] which in a sense is dual to those cited above. They consider general fluxes A, but only nondecreasing initial data u^0. If $A \in C^1(\mathbb{R})$, they again obtain (11). We do not know how the Total Oscillation Diminishing property relates to these results.

Throughout the paper, $\mathcal{S}(\mathbb{R}^n)$ means the usual Schwartz space.

2. Proof of the TOD property

In this section we prove the results given in Section 1. Since the hyperbolic TOD property follows from the parabolic one, we prove that first.

Proof of Theorem 1.4 We proceed in three steps. First we establish the result for strictly monotone smooth Φ, then we show how the general case can be reduced to this situation by suitable modification of Φ.

First step Let us assume first that Φ is smooth and $\left|\Phi'(u)\right| \geqslant \alpha$ for all $u \in \mathbb{R}$, where $\alpha > 0$ is some number. Since the solutions of (8) are smooth with fast decay at infinity, we can just compute as follows. We have

$$\frac{\partial}{\partial t}\Phi(u) + a(u)\frac{\partial}{\partial x}\Phi(u) = \Phi'(u)\frac{\partial^2 u}{\partial x^2},$$

and thus, setting $w = \frac{\partial}{\partial x}\Phi(u)$, we obtain

$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x}\big[a(u)w\big] = \frac{\partial}{\partial x}\left[\Phi'(u)\frac{\partial^2 u}{\partial x^2}\right].$$

Consider now a smooth convex function $G(w)$ such that $G(0) = 0$. Then we deduce from the previous equality

$$\frac{\partial}{\partial t}G(w) + \frac{\partial}{\partial x}\big[a(u)G(w)\big] + \frac{\partial}{\partial x}a(u)\ \big[wG'(w) - G(w)\big] = G'(w)\ \frac{\partial}{\partial x}\left[\Phi'(u)\frac{\partial^2 u}{\partial x^2}\right].$$

Next we integrate on the full line and get

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} G(w) + \int_{\mathbb{R}} \frac{\partial}{\partial x}a(u)\ \big[wG'(w) - G(w)\big] \\ = \int_{\mathbb{R}} G'(w)\ \frac{\partial}{\partial x}\left[\Phi'(u)\frac{\partial^2 u}{\partial x^2}\right] \\ = -\int_{\mathbb{R}} G''(w)\frac{\partial w}{\partial x}\ \Phi'(u)\frac{\partial^2 u}{\partial x^2} \end{aligned}$$

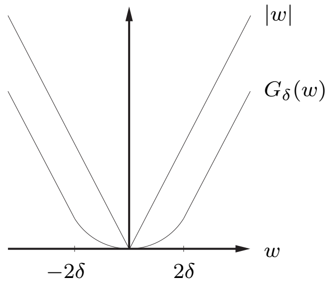

FIGURE 2. The function $G_\delta(w)$.

$$
\begin{aligned}
&= -\int_{\mathbb{R}} G''(w) \left[\Phi''(u)\Big(\frac{\partial u}{\partial x}\Big)^2 + \Phi'(u)\frac{\partial^2 u}{\partial x^2}\right]\Phi'(u)\frac{\partial^2 u}{\partial x^2}\\
&\leqslant -\int_{\mathbb{R}} G''(w)\ \Phi''(u)\Big(\frac{\partial u}{\partial x}\Big)^2\Phi'(u)\frac{\partial^2 u}{\partial x^2}\\
&= -\int_{\mathbb{R}} G''(w)w\ \Phi''(u)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}.
\end{aligned}
$$

To get rid of the two terms that control the time derivative we choose a convex function $G_\delta \geqslant 0$ such that $G_\delta(0) = 0$ and $G_\delta(w) = |w| - \delta$ for $|w| \geqslant 2\delta$, see Figure 2. Therefore $G_\delta(w) \to |w|$ and $wG'_\delta(w) - G_\delta(w) \to 0$ uniformly, as $\delta \downarrow 0$ (this is reminiscent to Kružkov's method [10] for BV estimates). Because of fast decay at infinity, the left-hand side term then satisfies

$$\int_{\mathbb{R}} \left|\frac{\partial}{\partial x}a(u)\right| \left|wG'_\delta(w) - G_\delta(w)\right| \longrightarrow 0.$$

The right-hand side term can be estimated as

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} G''_\delta(w)w\ \Phi''(u)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right|\\
&\qquad \leqslant C\int_{\{|w|\leqslant\delta\}} \left|\Phi''(u)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right| \leqslant C\delta\alpha^{-1}\int_{\mathbb{R}}\left|\Phi''(u)\frac{\partial^2 u}{\partial x^2}\right| \longrightarrow 0, \quad (12)
\end{aligned}
$$

where we used the fact that $w = \Phi'(u)\frac{\partial u}{\partial x}$ and $\left|\Phi'(u)\right| \geqslant \alpha$, by assumption. Recall that u is smooth with rapid decay at infinity, so the integral on the right-hand side of (12) is finite. Finally we have obtained

$$\frac{d}{dt}\int_{\mathbb{R}} |w| \leqslant 0,$$

which is the announced result of Theorem 1.4 for strictly monotone Φ.

Second step Assume now that Φ is Lipschitz continuous and monotone, say increasing. Then we consider a sequence of smooth functions Φ_α with $\Phi'_\alpha \geqslant \Phi' + \alpha$

and Φ'_α uniformly bounded, such that $\Phi'_\alpha \longrightarrow \Phi'$ a.e. as $\alpha \downarrow 0$. These Φ_α satisfy the assumptions of the first step. Therefore

$$\int_{\mathbb{R}} \left|\tfrac{\partial}{\partial x}\Phi_\alpha\big(u(t)\big)\right| dx \leqslant \int_{\mathbb{R}} \left|\tfrac{\partial}{\partial x}\Phi_\alpha\big(u^0\big)\right| dx, \quad \forall t > 0. \tag{13}$$

Note that the right-hand side is finite because Φ'_α is uniformly bounded and $u^0 \in \mathcal{S}(\mathbb{R})$. Consider now the limit $\alpha \downarrow 0$. Since u is smooth the left-hand side can be estimated from below using the chain rule

$$\left|\tfrac{\partial}{\partial x}\Phi_\alpha(u)\right| = \Phi'_a(u)\left|\tfrac{\partial u}{\partial x}\right| \geqslant \big(\Phi'(u)+\alpha\big)\left|\tfrac{\partial u}{\partial x}\right| \geqslant \left|\tfrac{\partial}{\partial x}\Phi(u)\right|.$$

On the right-hand side $\Phi'_\alpha \longrightarrow \Phi'$ a.e. and $u^0 \in \mathcal{S}(\mathbb{R})$ yield

$$\int_{\mathbb{R}} \left|\tfrac{\partial}{\partial x}\Phi_\alpha\big(u^0\big)\right| dx \longrightarrow \int_{\mathbb{R}} \left|\tfrac{\partial}{\partial x}\Phi\big(u^0\big)\right| dx.$$

This gives the result in the case of Lipschitz continuous monotone Φ.

Third step If Φ is an arbitrary Lipschitz continuous function, we decompose Φ canonically into its increasing and decreasing parts

$$\Phi = \Phi_+ + \Phi_- \quad \text{with} \quad \Phi'_+\Phi'_- = 0.$$

Then we apply the estimate we have just proved to the Lipschitz continuous monotone functions Φ_+ and Φ_- separately and use that for smooth u

$$\left|\tfrac{\partial}{\partial x}\Phi(u)\right| = \left|\tfrac{\partial}{\partial x}\Phi_+(u)\right| + \left|\tfrac{\partial}{\partial x}\Phi_-(u)\right| \tag{14}$$

since Φ'_+ and Φ'_- have disjoint support. This completes the proof. □

Proof of Proposition 1.2 Without loss of generality we assume that $\Phi(0) = 0$ because adding a constant to Φ does not change $|\Phi(u)|_{TV(\mathbb{R}^n)}$. By Lipschitz continuity, we then have $|\Phi(v)| \leqslant C|v|$ for $v \in \mathbb{R}$, and therefore $\Phi(u) \in L^1(\mathbb{R})$ whenever $u \in L^1(\mathbb{R})$. We consider a truncated and regularized approximation of the initial data u^0, i.e., $u^0_\delta \longrightarrow u^0$ in $L^1(\mathbb{R})$ as $\delta \to 0$. Let u_δ be the corresponding entropy solution to (1). This entropy solution is the unique limit as $\varepsilon \to 0$ of the parabolic approximation

$$\begin{cases} \frac{\partial}{\partial t}u_{\delta,\varepsilon} + \frac{\partial}{\partial x}A(u_{\delta,\varepsilon}) = \varepsilon\frac{\partial^2}{\partial x^2}u_{\delta,\varepsilon}, & t > 0,\ x \in \mathbb{R}, \\ u_{\delta,\varepsilon}(0,\cdot) = u^0_\delta \in \mathcal{S}(\mathbb{R}). \end{cases} \tag{15}$$

We now apply Theorem 1.4 and deduce that

$$\left|\Phi\big(u_{\delta,\varepsilon}(t)\big)\right|_{TV(\mathbb{R})} \leqslant \left|\Phi\big(u^0_\delta\big)\right|_{TV(\mathbb{R})}, \quad \forall t > 0.$$

As $\varepsilon \to 0$, we know that $u_{\delta,\varepsilon} \longrightarrow u_\delta$ in $C\big([0,T]; L^1(\mathbb{R})\big)$ for all $T > 0$, with a uniform L^∞ bound. Then also $\Phi(u_{\delta,\varepsilon}) \longrightarrow \Phi(u_\delta)$ and we obtain

$$\left|\Phi\big(u_\delta(t)\big)\right|_{TV(\mathbb{R})} \leqslant \left|\Phi\big(u^0_\delta\big)\right|_{TV(\mathbb{R})}, \quad \forall t > 0,$$

by lower semicontinuity of the total variation, see Proposition 3.6 in [1]. Now we may pass to the limit $\delta \to 0$. Then $u_\delta \longrightarrow u$ in $C(\mathbb{R}^+; L^1(\mathbb{R}))$, with a uniform L^∞ bound. This finally yields, by the same arguments,

$$\left|\Phi\big(u(t)\big)\right|_{TV(\mathbb{R})} \leqslant \liminf_{\delta\to 0} \left|\Phi\big(u_\delta^0\big)\right|_{TV(\mathbb{R})}, \quad \forall t > 0.$$

Since any initial family u_δ^0 gives rise to the same unique entropy limit, as a consequence of the $L^1(\mathbb{R})$–contraction, Proposition 1.2 is proved. □

Proof of Proposition 1.3 If $|\Phi(v)|_{TV(\mathbb{R}^n)} = \infty$, then any smooth approximation $v_\delta \longrightarrow v$ will do. Therefore we assume that $|\Phi(v)|_{TV(\mathbb{R}^n)} < \infty$. The difficulty then lies in the fact that we only have poor control over the total variation of v. We proceed in two steps. First we consider the case of a strictly monotone smooth Φ. Then we reduce the general case to the first one. Again we suppose $\Phi(0) = 0$. Then Lipschitz continuity of Φ implies that $\Phi(v)$ is absolutely integrable, and therefore $\Phi(v) \in BV(\mathbb{R}^n)$.

First step Assume that Φ is smooth and strictly increasing such that $\Phi'(u) \geqslant \alpha$ for all $u \in \mathbb{R}$, with $\alpha > 0$ some number. Following the proof of Theorem 3.9 in [1] we can find a sequence of $w_\delta \in C^\infty(\mathbb{R}^n)$ with $\operatorname{spt} w_\delta$ compact and $w_\delta \longrightarrow \Phi(v)$ in $L^1(\mathbb{R}^n)$ as $\delta \downarrow 0$, such that

$$\left|w_\delta\right|_{TV(\mathbb{R}^n)} \leqslant \left|\Phi(v)\right|_{TV(\mathbb{R}^n)} + \delta, \quad \forall \delta > 0.$$

Then we only need to prove that this approximation defines an approximation $v_\delta \in \mathcal{S}(\mathbb{R}^n)$ of v. But this follows from the strict monotonicity and smoothness of Φ: Simply put $v_\delta := \Phi^{-1}(w_\delta)$. Since Φ' is bounded away from zero, the Inverse Function Theorem implies that Φ^{-1} is smooth, in particular Lipschitz continuous. Hence v_δ is smooth by the chain rule and $v_\delta \longrightarrow v$ in $L^1(\mathbb{R}^n)$ because $\|v_\delta - v\|_{L^1(R^n)} \leqslant \alpha^{-1}\|w_\delta - \Phi(v)\|_{L^1(\mathbb{R}^n)}$. Each v_δ has compact support since $\operatorname{spt} w_\delta$ is compact and $\Phi(0) = 0$.

Second step Assume now that Φ is merely Lipschitz continuous and increasing. We will show that for any $\delta > 0$ there exists $v_\delta \in \mathcal{S}(\mathbb{R}^n)$ with

$$\|v_\delta - v\|_{L^1(\mathbb{R}^n)} < 4\delta \quad \text{and} \quad \left|\Phi(v_\delta)\right|_{TV(\mathbb{R}^n)} \leqslant \left|\Phi(v)\right|_{TV(\mathbb{R}^n)} + 3\delta.$$

We proceed as follows: For given $\delta > 0$ we first construct a $\tilde{v} \in BV(\mathbb{R}^n)$ for which we still control the total variation of $\Phi(\tilde{v})$. Then we replace Φ by a strictly monotone smooth Φ_α. The total variation of $\Phi_\alpha(\tilde{v})$ is bounded since the error term can be estimated against the $BV(\mathbb{R}^n)$–norm of $\tilde{v}$. Then we apply the argument of the first step. For simplicity we assume that $v \geqslant 0$. The general case can be proved by applying the following arguments to the positive and negative parts of v separately.

We first construct a function $\hat{v}$ such that $\operatorname{spt}\hat{v}$ has finite measure with

$$\|\hat{v} - v\|_{L^1(\mathbb{R}^n)} \leqslant \delta \quad \text{and} \quad \left|\Phi(\hat{v})\right|_{TV(\mathbb{R}^n)} \leqslant \left|\Phi(v)\right|_{TV(\mathbb{R}^n)} + \delta,$$

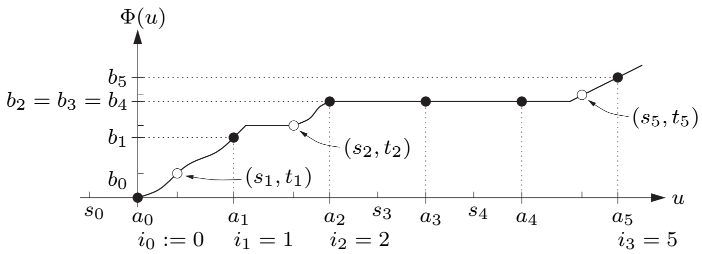

FIGURE 3. Approximation of $\Phi(u)$.

By the coarea formula for $BV(\mathbb{R}^n)$–functions, see Theorem 3.40 of [1],

$$\left|\Phi(v)\right|_{TV(\mathbb{R}^n)} = \int_{-\infty}^{\infty} P_{\Phi(v)}(t)\, dt,$$

where $P_{\Phi(v)}(t)$ is the perimeter of the level set $\{\Phi(v) > t\}$, i.e.,

$$P_{\Phi(v)}(t) := \left|\mathbf{1}_{\{\Phi(v)>t\}}\right|_{TV(\mathbb{R}^n)}.$$

Since $\Phi(v) \in L^\infty \cap BV(\mathbb{R}^n)$, $P_{\Phi(v)}$ has compact support and is finite a.e. Moreover, $P_{\Phi(v)}(t) = 0$ for $t < 0$ since $v \geqslant 0$. If Φ is not strictly increasing in 0, by continuity we can find $\beta > 0$ such that $\Phi([0, \beta]) = \{0\}$ and $\int_{\{v \leqslant \beta\}} v(x)\, dx \leqslant \delta$. Therefore, if we let $\hat{v} := v\mathbf{1}_{\{\beta<v\}}$, then $\operatorname{spt} \hat{v}$ has finite measure by the Chebyshev inequality and it holds

$$\|\hat{v} - v\|_{L^1(\mathbb{R}^n)} \leqslant \delta \quad \text{and} \quad \left|\Phi(\hat{v})\right|_{TV(\mathbb{R}^n)} = \left|\Phi(v)\right|_{TV(\mathbb{R}^n)}.$$

Otherwise, if Φ is strictly increasing in 0, then we can find $\beta > 0$ such that

$$\max\left\{\int_{\{\Phi(v)\leqslant\beta\}} v(x)\, dx,\ \int_{-\infty}^{\beta} P_{\Phi(v)}(t)\, dt\right\} \leqslant \delta.$$

We fix a Lebesgue point $0 < \bar{t} < \beta$ of $P_{\Phi(v)}$ with

$$\beta P_{\Phi(v)}(\bar{t}) \leqslant \int_{-\infty}^{\beta} P_{\Phi(v)}(t)\, dt,$$

and define $\hat{v} := v\mathbf{1}_{\{\bar{t}<\Phi(v)\}}$. Since Φ is increasing, $\operatorname{spt} \hat{v}$ has finite measure by the Chebyshev inequality and we get $\|\hat{v} - v\|_{L^1(\mathbb{R}^n)} \leqslant \delta$ and

$$\left|\Phi(\hat{v})\right|_{TV(\mathbb{R}^n)} \leqslant \int_{\bar{t}}^{\infty} P_{\Phi(v)}(t)\, dt + \bar{t} P_{\Phi(v)}(\bar{t}) \leqslant \left|\Phi(v)\right|_{TV(\mathbb{R}^n)} + \delta.$$

For $0 < \gamma < \delta/|\operatorname{spt} \hat{v}|$ let $a_i := \gamma i$ and $b_i := \Phi(a_i)$ for all $i \in \mathbb{Z}$. Starting from $i_0 := 0$ define recursively

$$i_{\nu+1} := \min\left\{ i > i_\nu \,\middle|\, b_i > b_{i_\nu} \text{ and } a_{i_\nu} \leqslant \gamma + \|\hat{v}\|_{L^\infty(\mathbb{R}^n)} \right\}$$

see Figure 3. Since $\hat{v}$ is bounded, we may assume that Φ is growing to infinity and then there are only finitely many such i_ν, say $\Lambda := \{i_1, \ldots, i_N\}$. By definition,

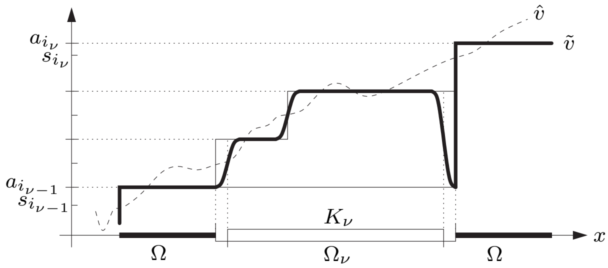

FIGURE 4. Approximation of $\hat{v}$.

$b_{i_{\nu-1}} = b_{i_{\nu-1}+1} = \cdots = b_{i_\nu - 1}$. For each $i_\nu \in \Lambda$ we pick a number $t_{i_\nu} \in (b_{i_\nu - 1}, b_{i_\nu})$ such that

$$(b_{i_\nu} - b_{i_\nu - 1})\, P_{\Phi(\hat v)}(t_{i_\nu}) \leqslant \int_{b_{i_\nu - 1}}^{b_{i_\nu}} P_{\Phi(\hat v)}(t)\, dt. \tag{16}$$

If $s_{i_\nu} := \sup\{s \in \mathbb{R} \mid \Phi(s) \leqslant t_{i_\nu}\}$, then $\{\Phi(v) > t_{i_\nu}\} = \{v > s_{i_\nu}\}$, see Figure 3. Since t_{i_ν} is in the interior of the open interval $(b_{i_\nu - 1}, b_{i_\nu})$, we have $s_{i_\nu} \in (a_{i_\nu - 1}, a_{i_\nu})$. Note also that $s_{i_N} > \|\hat v\|_{L^\infty(\mathbb{R})}$. For $k \in \mathbb{Z}\setminus\Lambda$ we define $s_k := (a_{k-1} + a_k)/2$. Consider then the simple function

$$\tilde v := \sum_{\nu=1}^{N} a_{i_{\nu-1}}\, \mathbf{1}_{\{s_{i_\nu} \geqslant \hat v > s_{i_{\nu-1}}\}},$$

see Figure 4. Then also $\Phi(\tilde v)$ is a simple function and we can estimate

$$\begin{aligned}
|\Phi(\tilde v)|_{TV(\mathbb{R}^n)} &= \left| \sum_{\nu=1}^{N} b_{i_{\nu-1}}\, \mathbf{1}_{\{s_{i_\nu} \geqslant \hat v > s_{i_{\nu-1}}\}} \right|_{TV(\mathbb{R}^n)} \\
&= \sum_{\nu=1}^{N-1} (b_{i_\nu} - b_{i_{\nu-1}})\, \left|\mathbf{1}_{\{\hat v > s_{i_\nu}\}}\right|_{TV(\mathbb{R}^n)} \\
&= \sum_{\nu=1}^{N-1} (b_{i_\nu} - b_{i_\nu - 1})\, P_{\Phi(\hat v)}(t_{i_\nu}) \\
&\leqslant \sum_{\nu=1}^{N-1} \int_{b_{i_\nu - 1}}^{b_{i_\nu}} P_{\Phi(\hat v)}(t)\, dt \leqslant |\Phi(\hat v)|_{TV(\mathbb{R}^n)},
\end{aligned} \tag{17}$$

using $b_{i_\nu - 1} = b_{i_{\nu-1}}$, (16) and the coarea formula. Since $b_{i_\nu} - b_{i_{\nu-1}} > 0$ and N is finite, we also conclude from (17) that

$$|\tilde v|_{TV(\mathbb{R}^n)} = \sum_{\nu=1}^{N-1} (a_{i_\nu} - a_{i_{\nu-1}})\, \left|\mathbf{1}_{\{\hat v > s_{i_\nu}\}}\right|_{TV(\mathbb{R}^n)}$$

$$\leqslant \quad \frac{2\|\hat{v}\|_{L^\infty(\mathbb{R}^n)}}{\min\limits_{\nu=1,\dots,N-1}\big(b_{i_\nu}-b_{i_\nu-1}\big)}\,|\Phi(\hat{v})|_{TV(\mathbb{R}^n)} < \infty.$$

Then $\tilde{v} \in BV(\mathbb{R}^n)$ because $\operatorname{spt}\tilde{v} \subset \operatorname{spt}\hat{v}$ has finite measure. Consider

$$\Omega := \bigcup \big\{ s_{i_{\nu-1}+1} \geqslant \hat{v} > s_{i_{\nu-1}} \,\big|\, \nu = 1,\dots,N \big\}.$$

Since $a_{i_{\nu-1}-1} < s_{i_{\nu-1}} < a_{i_{\nu-1}} < s_{i_{\nu-1}+1} < a_{i_{\nu-1}+1}$ we have

$$\begin{aligned} \|\tilde{v}-\hat{v}\|_{L^1(\Omega)} &= \sum_{\nu=1}^{N} \int_{\{s_{i_{\nu-1}+1}\geqslant \hat{v}>s_{i_{\nu-1}}\}} \big|\hat{v}(x)-a_{i_{\nu-1}}\big|\,dx \\ &\leqslant \gamma \sum_{\nu=1}^{N} \big|\big\{s_{i_{\nu-1}+1} \geqslant \hat{v} > s_{i_{\nu-1}}\big\}\big| = |\Omega|\gamma \end{aligned} \tag{18}$$

which is small by our choice of γ. Therefore $\tilde{v}$ is close to $\hat{v}$ in the set Ω.

We are now going to modify $\tilde{v}$ in $\mathbb{R}^n\backslash\Omega$ in such a way that the new function (again denoted by $\tilde{v}$) satisfies the estimates

$$\|\tilde{v}-\hat{v}\|_{L^1(\mathbb{R}^n)} < \delta \quad \text{and} \quad |\Phi(\tilde{v})|_{TV(\mathbb{R}^n)} \leqslant |\Phi(\hat{v})|_{TV(\mathbb{R}^n)}.$$

Note first that Φ is constant on $[a_{i_{\nu-1}}, a_{i_\nu-1}]$ for all $\nu = 1,\dots,N$. Therefore we consider the set $\Omega_\nu := \{s_{i_\nu} \geqslant \hat{v} > s_{i_{\nu-1}+1}\}$. If $\Omega_\nu = \emptyset$, there is nothing to do. Otherwise we redefine $\tilde{v}$ in Ω_ν as

$$\tilde{v} := \sum_{k=i_{\nu-1}+2}^{i_\nu} a_{k-1}\,\mathbf{1}_{\{s_k\geqslant\hat{v}>s_{k-1}\}}, \tag{19}$$

see Figure 4. The same argument that led to (18) then shows that

$$\|\tilde{v}-\hat{v}\|_{L^1(\Omega_\nu)} \leqslant |\Omega_\nu|\gamma,$$

and summing over ν and using (18) yields $\|\tilde{v}-\hat{v}\|_{L^1(\mathbb{R}^n)} \leqslant \delta$ because $\gamma < \delta/|\operatorname{spt}\hat{v}|$. Moreover, $|\Phi(\tilde{v})|_{TV(\mathbb{R}^n)} \leqslant |\Phi(\hat{v})|_{TV(\mathbb{R}^n)}$ by (17) because inserting the extra states (19) does not change $\Phi(\tilde{v})$.

One final modification of $\tilde{v}$ is needed to get a function in $BV(\mathbb{R}^n)$. For each $\nu = 1,\dots,N$ we will therefore approximate $\tilde{v}$ in $L^1(\Omega_\nu)$ by a smooth function which takes values in $[a_{i_{\nu-1}}, a_{i_\nu-1}]$ only. Again this will not change $|\Phi(\tilde{v})|_{TV(\mathbb{R}^n)}$. We proceed as follows: We first redefine $\tilde{v}$ by $a_{i_{\nu-1}}$ in $\Omega_\nu\backslash K_\nu$ with $K_\nu \subset \operatorname{int}\Omega_\nu$ compact and then convolve with a standard mollifier ρ_ν with $\operatorname{diam}\operatorname{spt}\rho_\nu < \operatorname{dist}(K_\nu, \mathbb{R}^n\backslash\Omega_\nu)$, see Figure 4. By choosing K_ν and ρ_ν appropriately, the induced $L^1(\Omega_\nu)$–error does not exceed δ/N. Repeating this for all ν gives $\tilde{v} \in BV(\mathbb{R}^n)$.

As in the proof of Theorem 1.4 we now consider a sequence of smooth Φ_α with $\Phi'_\alpha \geqslant \Phi' + \alpha$ and Φ'_α uniformly bounded, such that $\Phi_\alpha \longrightarrow \Phi$ a.e. as $\alpha \downarrow 0$. Since $\tilde{v} \in BV(\mathbb{R}^n)$, we can find α small enough such that

$$|\Phi_\alpha(\tilde{v})|_{TV(\mathbb{R}^n)} \leqslant |\Phi(\tilde{v})|_{TV(\mathbb{R}^n)} + \delta.$$

Then we conclude as in the first step, applying the argument to Φ_α and $\tilde{v}$ instead of Φ and v. Since v_δ is smooth, the chain rule gives

$$\begin{aligned} |\Phi(v_\delta)|_{TV(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \Phi'(v_\delta)|Dv_\delta(x)|\,dx \\ &\leqslant \int_{\mathbb{R}^n} \Phi'_\alpha(v_\delta)|Dv_\delta(x)|\,dx = |w_\delta|_{TV(\mathbb{R}^n)}, \end{aligned}$$

where w_δ is the approximation of $\Phi_\alpha(\tilde{v})$ and $v_\delta := \Phi_\alpha^{-1}(w_\delta)$. □

3. Semi-discrete TOD schemes

In this section, we consider a TOD property for semi-discrete approximations of the conservation law (1). Namely, we consider schemes

$$\begin{cases} \frac{d}{dt}v_i(t) + \frac{1}{\Delta x_i}\big[A_{i+1/2}(t) - A_{i-1/2}(t)\big] = 0, \\ A_{i+1/2}(t) := \mathcal{A}\big(v_i(t), v_{i+1}(t)\big) \quad \text{(3-point scheme)} \end{cases}$$

(see [7], [12] for the status of this theory as well as for historical and more advanced references). As usual we require some regularity for $\mathcal{A}$ such as Lipschitz continuity, and we assume that the scheme is consistent, i.e.,

$$\mathcal{A}(u,u) = A(u), \quad \forall u \in \mathbb{R}.$$

We recall the following basic notions of stability (see [12])

Definition 3.1. A consistent discrete flux $\mathcal{A}(\cdot,\cdot)$ is called

(i) monotone if

$$\frac{\partial}{\partial u}\mathcal{A}(u,v) \geqslant 0, \quad \frac{\partial}{\partial v}\mathcal{A}(u,v) \leqslant 0.$$

(ii) E-flux if

$$\begin{cases} \mathcal{A}(u,v) \leqslant A(\xi) & \text{for } u \leqslant \xi \leqslant v, \\ \mathcal{A}(u,v) \geqslant A(\xi) & \text{for } v \leqslant \xi \leqslant u. \end{cases}$$

(iii) ordered if

$$\begin{cases} \mathcal{A}(u,v) \leqslant \min(A(u),A(v)) & \text{for } u \leqslant v, \\ \mathcal{A}(u,v) \geqslant \max(A(u),A(v)) & \text{for } v \leqslant u. \end{cases}$$

The known properties being summarized as follows

monotone discrete flux	$\Longleftrightarrow$	contraction (multidimensional)
$\Downarrow$		
E-discrete flux	$\Longleftrightarrow$	discrete entropy inequalities (multidimensional)
$\Downarrow$		
ordered discrete flux	$\Longleftrightarrow$	TVD (onedimensional)

We can complete this picture by the

Proposition 3.2 (Semi-discrete TOD schemes)**.** *Any semi-discrete ordered scheme is TOD, i.e., for all Lipschitz continuous monotone functions* Φ

$$\sum_{i\in\mathbb{Z}} \left|\Phi\big(v_{i+1}(t)\big) - \Phi\big(v_i(t)\big)\right| \leqslant \sum_{i\in\mathbb{Z}} \left|\Phi\big(v_{i+1}^0\big) - \Phi\big(v_i^0\big)\right|, \quad \forall t > 0.$$

The assumption of monotonicity of Φ *cannot be dropped.*

Proof. We consider the positive part $\big[\Phi(v_{i+1}(t)) - \Phi(v_i(t))\big]_+$ rather than the absolute value to simplify the proof somewhat. This is equivalent because the scheme is conservative and for all $a \in \mathbb{R}$ it holds $|a| = 2a_+ - a$. We also consider only the case of increasing Φ. We start by writing

$$\frac{d}{dt}\Phi\big(v_i(t)\big) + \frac{\Phi'\big(v_i(t)\big)}{\Delta x_i}\Big[A_{i+1/2}(t) - A_{i-1/2}(t)\Big] = 0,$$

which implies

$$\begin{aligned}&\frac{d}{dt}\Big[\Phi\big(v_{i+1}(t)\big) - \Phi\big(v_i(t)\big)\Big]_+ \\ &+ \mathbf{1}_{\left\{\Phi(v_{i+1}(t))\geqslant\Phi(v_i(t))\right\}}\left[\frac{\Phi'\big(v_{i+1}(t)\big)}{\Delta x_{i+1}}\Big(A_{i+3/2}(t) - A_{i+1/2}(t)\Big)\right.\\ &\qquad\qquad\left. - \frac{\Phi'\big(v_i(t)\big)}{\Delta x_i}\Big(A_{i+1/2}(t) - A_{i-1/2}(t)\Big)\right] = 0.\end{aligned}$$

Summing up this equality yields

$$\begin{aligned}&\frac{d}{dt}\sum_{i\in\mathbb{Z}}\Big[\Phi\big(v_{i+1}(t)\big) - \Phi\big(v_i(t)\big)\Big]_+ \\ &+ \sum_{i\in\mathbb{Z}}\frac{\Phi'\big(v_i(t)\big)}{\Delta x_i}\Big(A_{i+1/2}(t) - A_{i-1/2}(t)\Big)\\ &\qquad\qquad\Big[\mathbf{1}_{\left\{\Phi(v_i(t))\geqslant\Phi(v_{i-1}(t))\right\}} - \mathbf{1}_{\left\{\Phi(v_{i+1}(t))\geqslant\Phi(v_i(t))\right\}}\Big] = 0. \quad (20)\end{aligned}$$

Note that the second sum in (20) contains only those sites i where $\Phi(v_i)$ has a strict local maximum or minimum. This corresponds to the continuous case where an integration over the set $\{w = 0\}$ with $w = \frac{\partial}{\partial x}\Phi(u)$ was involved, see (12). It remains to prove that for all $i \in \mathbb{Z}$ we have

$$\begin{aligned}
&\Big(A_{i+1/2}(t) - A_{i-1/2}(t)\Big) \\
&\qquad\qquad \Big[\mathbf{1}_{\big\{\Phi(v_i(t))\geqslant\Phi(v_{i-1}(t))\big\}} - \mathbf{1}_{\big\{\Phi(v_{i+1}(t))\geqslant\Phi(v_i(t))\big\}}\Big] \\
&\quad = \Big(A_{i+1/2}(t) - A_{i-1/2}(t)\Big) \\
&\qquad\qquad\qquad \Big[\mathbf{1}_{\big\{\Phi(v_i(t))\geqslant\Phi(v_{i\pm1}(t))\big\}} - \mathbf{1}_{\big\{\Phi(v_i(t))\leqslant\Phi(v_{i\pm1}(t))\big\}}\Big] \geqslant 0.
\end{aligned}$$

Consider for instance the term with the *plus* sign. Since Φ is increasing

$$\begin{aligned}
&\Big(A_{i+1/2}(t) - A_{i-1/2}(t)\Big)\,\mathbf{1}_{\big\{\Phi(v_i(t))\geqslant\Phi(v_{i\pm1}(t))\big\}} \\
&\quad = \Big[\mathcal{A}\big(v_i(t), v_{i+1}(t)\big) - \mathcal{A}\big(v_{i-1}(t), v_i(t)\big)\Big]\,\mathbf{1}_{\big\{v_i(t)\geqslant v_{i\pm1}(t)\big\}} \\
&\quad = \Big[\mathcal{A}\big(v_i(t), v_{i+1}(t)\big) - A\big(v_i(t)\big)\Big]\,\mathbf{1}_{\big\{v_i(t)\geqslant v_{i\pm1}(t)\big\}} \\
&\qquad\qquad + \Big[A\big(v_i(t)\big) - \mathcal{A}\big(v_{i-1}(t), v_i(t)\big)\Big]\,\mathbf{1}_{\big\{v_i(t)\geqslant v_{i\pm1}(t)\big\}}
\end{aligned}$$

and both terms are positive under the "ordered" assumption (and only if it holds). Similarly, for the term with *minus* sign we have

$$\begin{aligned}
&-\Big(A_{i+1/2}(t) - A_{i-1/2}(t)\Big)\,\mathbf{1}_{\big\{\Phi(v_i(t))\leqslant\Phi(v_{i\pm1}(t))\big\}} \\
&\quad = -\Big[\mathcal{A}\big(v_i(t), v_{i+1}(t)\big) - \mathcal{A}\big(v_{i-1}(t), v_i(t)\big)\Big]\,\mathbf{1}_{\big\{v_i(t)\leqslant v_{i\pm1}(t)\big\}} \\
&\quad = -\Big[\mathcal{A}\big(v_i(t), v_{i+1}(t)\big) - A\big(v_i(t)\big)\Big]\,\mathbf{1}_{\big\{v_i(t)\leqslant v_{i\pm1}(t)\big\}} \\
&\qquad\qquad - \Big[A\big(v_i(t)\big) - \mathcal{A}\big(v_{i-1}(t), v_i(t)\big)\Big]\,\mathbf{1}_{\big\{v_i(t)\leqslant v_{i\pm1}(t)\big\}}
\end{aligned}$$

and both terms are again positive under the "ordered" assumption. This proves that semi-discrete ordered schemes have the TOD property.

We conclude this section by proving that monotonicity of Φ is indispensable. To this end, fix some $v \in \mathbb{R}$ which is not a maximum of A. We assume that A is not constant. Then there is a $u \neq v$ with

$$\begin{cases} \text{either} & \mathcal{A}(v,u) \neq A(v) \quad \text{if } u < v \\ \text{or} & \mathcal{A}(u,v) \neq A(v) \quad \text{if } v < u. \end{cases}$$

In fact, assume that not. Then Definition 3.1 (iii) implies

$$\begin{cases} \mathcal{A}(v,u) = A(v) \geqslant A(u) & \text{for all } u < v \\ \mathcal{A}(u,v) = A(v) \geqslant A(u) & \text{for all } v < u, \end{cases}$$

in contradiction to our choice of v. For definiteness let us assume that $u < v$ with $\mathcal{A}(v,u) > A(v)$. The other cases can be handled in a similar way. By Lipschitz continuity and consistency $\mathcal{A}(u,u) = A(u)$ we can find $w > v$ such that $\mathcal{A}(v,u) > \mathcal{A}(w,v)$. Now we define initial data

$$v_0(0) := v \quad \text{and} \quad v_i(0) := \begin{cases} w & \text{for } i < 0 \\ u & \text{for } i > 0. \end{cases} \tag{21}$$

Consider equation (20) for $t = 0$. We can find Φ such that

$$\Phi(u) = \Phi(v), \quad \Phi'(v) > 0 \quad \text{and} \quad \Phi'(w) = 0.$$

In fact, a suitable cubic polynomial will do. By construction, we then have

$$\begin{aligned} &\frac{d}{dt} \sum_{i \in \mathbb{Z}} \Big[\Phi\big(v_{i+1}(0)\big) - \Phi\big(v_i(0)\big) \Big]_+ \qquad (22) \\ &= - \sum_{i \in \mathbb{Z}} \frac{\Phi'\big(v_i(t)\big)}{\Delta x_i} \Big(A_{i+1/2}(t) - A_{i-1/2}(t) \Big) \\ &\qquad \Big[\mathbf{1}_{\{\Phi(v_i(t)) \geqslant \Phi(v_{i-1}(t))\}} - \mathbf{1}_{\{\Phi(v_{i+1}(t)) \geqslant \Phi(v_i(t))\}} \Big] \\ &= - \frac{\Phi'(v)}{\Delta x_0} \Big(\mathcal{A}(v,u) - \mathcal{A}(w,v) \Big) > 0. \end{aligned}$$

This proves that the seminorm must grow. □

Remark 3.3. In Proposition 1.2 we had to put a lim inf in the initial control to cope with nonmonotone Φ. No analogue of this procedure exists at the semi-discrete level. Note that the initial data we defined in (21) approximates a Riemann problem, i.e., a discontinuity. Then the time derivative in (22) becomes infinite as $\Delta x_i \to 0$, and a kind of "boundary layer" is established: the seminorm is instantaneously increased to the value which supposedly corresponds to the lim inf in the continuous setting.

4. Fully discrete TOD schemes

Since it is not obvious how to prove the TOD property for a general fully discrete scheme, we only consider the classical Godunov scheme. For simplicity, we also restrict to the case of a regular grid. Let us quickly recall the definition of the Godunov scheme: Given $h > 0$ we define intervals $I_i := [ih, (i+1)h]$ for $i \in \mathbb{Z}$ and then introduce the projection operator

$$P_h u(x) := h^{-1} \int_{I_i} u(y)\, dy \quad \text{for } x \in I_i.$$

The numerical solution obtained by the Godunov scheme is piecewise constant in time and space, and is therefore determined by the value v_i^n it takes in the interval I_i at discrete times t^n to be specified in a moment. Therefore we consider $v^n = \sum_{i\in\mathbb{Z}} v_i^n \mathbf{1}_{I_i}$ for $n \geqslant 0$. Given data $u^0 \in L^1 \cap L^\infty(\mathbb{R})$ we define $v^0 := P_h u^0$ and then use the update formula

$$v^n := P_h u^n\big(\Delta t^n, \cdot\big), \quad \forall n > 0,$$

where u^n is the unique entropy solution to

$$\begin{cases} \frac{\partial}{\partial t} u^n + \frac{\partial}{\partial x} A(u^n) = 0, & t > 0,\ x \in \mathbb{R}, \\ u^n(0,\cdot) = v^{n-1}, \end{cases} \tag{23}$$

and the time steps $\Delta t^n := t^n - t^{n-1}$ are chosen in such a way that the CFL condition $\Delta t^n h^{-1} \|a\|_{L^\infty(\mathbb{R})} \leqslant \lambda$ for some $\lambda \leqslant 1/2$ is satisfied. Then an explicit formula for the numerical flux is available, see [7].

We have the following result.

Proposition 4.1. *The Godunov scheme is TOD, i.e., for all Lipschitz continuous monotone functions* Φ

$$\sum_{i\in\mathbb{Z}} \big|\Phi\big(v_{i+1}^n\big) - \Phi\big(v_i^n\big)\big| \leqslant \sum_{i\in\mathbb{Z}} \big|\Phi\big(v_{i+1}^0\big) - \Phi\big(v_i^0\big)\big|, \quad \forall n > 0.$$

Proof. Since v^n is piecewise constant, the same is true for $\Phi(v^n)$. Then

$$\big|\Phi(v^n)\big|_{TV(\mathbb{R})} = \sum_{i\in\mathbb{Z}} \big|\Phi\big(v_{i+1}^n\big) - \Phi\big(v_i^n\big)\big|, \quad \forall n \geqslant 0.$$

We already know by Theorem 1.1 that the solution operator of (23) is Total Oscillation Diminishing. Therefore it suffices to prove the following

Claim: For all $v \in L^1 \cap L^\infty(\mathbb{R})$

$$\big|\Phi\big(P_h v\big)\big|_{TV(\mathbb{R})} \leqslant \big|\Phi(v)\big|_{TV(\mathbb{R})}. \tag{24}$$

Again we assume that $\Phi(0) = 0$. Consider functions $v_\delta \in \mathcal{S}(\mathbb{R})$ with $v_\delta \longrightarrow v$ in $L^1(\mathbb{R})$ and $\limsup_{\delta\to 0} |\Phi(v_\delta)|_{TV(\mathbb{R})} \leqslant |\Phi(v)|_{TV(\mathbb{R})}$, as in Proposition 1.3. Since P_h is $L^1(\mathbb{R})$–continuous, the Lipschitz continuity of Φ implies $\Phi(P_h v_\delta) \longrightarrow \Phi(P_h v)$ in $L^1(\mathbb{R})$, and then

$$\big|\Phi\big(P_h v\big)\big|_{TV(\mathbb{R})} \leqslant \liminf_{\delta\to 0} \big|\Phi\big(P_h v_\delta\big)\big|_{TV(\mathbb{R})}$$

by lower semicontinuity of the total variation, see Proposition 3.6 in [1]. Therefore we only need to prove (24) for $v \in \mathcal{S}(\mathbb{R})$. We slightly modify the argument of

Lemma 8 in [9]. For $\lambda \in \mathbb{R}$ let $E_\lambda := \{x \mid v(x) > \lambda\}$. Since v is bounded, we have $E_\lambda(v) = \emptyset$ for large λ. Then also $v_j := P_h v|_{I_j} \leqslant \lambda$ for all $j \in \mathbb{Z}$, and thus $E_\lambda(P_h v)$ is empty as well. If $E_\lambda(v) \neq \emptyset$, then it can be decomposed into at most countably many connected components

$$E_\lambda(v) = \bigcup_{\ell=1}^{L_\lambda} O_\ell \quad \text{for some } L_\lambda,$$

where O_ℓ are disjoint open intervals. Note that $E_\lambda(v)$ is open because v is smooth. We will now show that $E_\lambda(P_h v)$ has at most L_λ connected components. There is nothing to do if $L_\lambda = \infty$. Otherwise we observe that

(a) If $\operatorname{int} I_j \subset E_\lambda(v)$, then $v_j > \lambda$ and $\operatorname{int} I_j \subset E_\lambda(P_h v)$.
(b) If $I_j \cap E_\lambda(v) = \emptyset$, then $v_j \leqslant \lambda$ and $I_j \cap E_\lambda(P_h v) = \emptyset$.

For each I_j we either have (a) or (b), or I_j is an end interval, i.e., $I_j \cap O_\ell \neq \emptyset$ and $I_j \not\subset O_\ell$ for some ℓ. Let $J_\ell := \{j \mid \operatorname{int} I_j \subset E_\lambda(P_h v) \text{ and } I_j \cap O_\ell \neq \emptyset\}$ and $O_\ell^* := \operatorname{int} \bigcup_{j \in J_\ell} I_j$. Then O_ℓ^* contains all I_j that were already present in O_ℓ (because of (a)) plus possibly some end intervals of O_ℓ. Therefore O_ℓ^* is either empty or an open interval. On the other hand, $E_\lambda(P_h v) \subset \bigcup_\ell O_\ell^*$ since (b) implies that no $I_j \in E_\lambda(P_h v)$ can exist which hits none of the O_ℓ. Let L_λ^* be the number of connected components of $E_\lambda(P_h v)$. There can be at most as many of these as there are O_ℓ^*, thus $L_\lambda^* \leqslant L_\lambda$.

Now note that the perimeters of the level sets $E_\lambda(v)$ and $E_\lambda(P_h v)$ equal just $2L_\lambda$ and $2L_\lambda^*$, respectively. Therefore we can use the coarea formula and the chain rule once more to estimate

$$\begin{aligned} \big|\Phi\big(P_h v\big)\big|_{TV(R)} &= 2 \int_{\mathbb{R}} \Phi'(\lambda)\, L_\lambda^* \, d\lambda \\ &\leqslant 2 \int_{\mathbb{R}} \Phi'(\lambda)\, L_\lambda \, d\lambda = \big|\Phi(v)\big|_{TV(R)}. \end{aligned}$$

This proves the claim and the proposition. □

Remark 4.2. It would be interesting to understand, for other schemes, which CFL condition enforces the TOD property. As discussed in Section 1 the Lax-Friedrichs scheme is TVD, but not TOD for certain timesteps. Therefore the two notions seem to use different CFL conditions. This is in contrast to the semi-discrete case because the diagram on page 13 and Proposition 3.2 together imply that for semi-discrete schemes TVD and TOD are in fact equivalent.

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