THE POLAR CONE OF THE SET OF MONOTONE MAPS

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Abstract. We prove that every element of the polar cone to the closed convex cone of monotone transport maps can be represented as the divergence of a measure field taking values in the positive definite matrices.

Contents

1. Introduction

The one-dimensional pressureless gas dynamics equations

$$\left.\begin{aligned} \partial_t \varrho + \nabla\cdot(\varrho \boldsymbol{u}) &= 0 \\ \partial_t(\varrho\boldsymbol{u}) + \nabla\cdot(\varrho\boldsymbol{u}\otimes\boldsymbol{u}) &= 0 \end{aligned}\right\} \quad \text{in } [0,\infty)\times\mathbb{R} \tag{1.1}$$

has recently been shown equivalent (in the regime of sticky particles) to a first-order differential inclusion on the space of monotone transport maps from the reference measure space $([0,1], \mathcal{L}^1|_{[0,1]|}) =: (\Omega, \mathsf{m})$ (where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure) to $\mathbb{R}$; see [7]. More precisely, to every density/velocity $(\varrho, \boldsymbol{u})$ solving (1.1) one can associate a unique map $X \in \mathscr{L}^2(\Omega, \mathsf{m})$ with X *monotone* such that

$$\varrho(t,\cdot) = X(t,\cdot)\#\mathsf{m} \quad \text{for all } t\in[0,\infty). \tag{1.2}$$

Here # indicates the push-forward of measures. Then X satisfies

$$\dot{X} + \partial I_{\mathscr{K}}(X) \ni \bar{V} \quad \text{for all } t\in[0,\infty), \tag{1.3}$$

where $\mathscr{K}$ denotes the closed convex cone of all transport maps $X \in \mathscr{L}^2(\Omega,\mathsf{m})$ that are monotone, and where $\partial I_{\mathscr{K}}$ is the subdifferential of the indicator function of $\mathscr{K}$. If X satisfies (1.3) and is related to ϱ through (1.2), then the Eulerian velocity $\boldsymbol{u}$ can be recovered from the Lagrangian velocity $V := \dot{X}$ through

$$V(t,\cdot) = \boldsymbol{u}(t, X(t,\cdot)) \quad \text{for all } t\in[0,\infty). \tag{1.4}$$

Assuming finite kinetic energy, it is natural to require that

$$V(t,\cdot)\in\mathscr{L}^2(\Omega,\mathsf{m}), \quad \boldsymbol{u}(t,\cdot)\in\mathscr{L}^2(\mathbb{R},\varrho(t,\cdot)).$$

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The relation (1.4) in particular determines the initial Lagrangian velocity $\bar{V}$ in (1.3) in terms of the initial data $(\varrho, \boldsymbol{u})(0,\cdot) =: (\bar{\varrho}, \bar{\boldsymbol{u}})$ of the system (1.1).

It is shown in [7] that the solution of (1.3) can be written explicitly as

$$X(t,\cdot) = P_{\mathscr{K}}(\bar{X} + t\bar{V}) \quad \text{for all } t \in [0,\infty), \tag{1.5}$$

with $\bar{X} := X(0,\cdot) \in \mathscr{K}$ given by (1.2). Here $P_{\mathscr{K}}$ denotes the metric projection onto the cone $\mathscr{K}$. The connection between (1.1) and (1.3) makes it possible to apply classical results from the theory of first-order differential inclusions in Hilbert spaces to study the pressureless gas dynamics equations, which form a system of hyperbolic conservation laws. We refer the reader to [3,7] for further information.

It is known that if X satisfies (1.5), then the difference $(\bar{X} + t\bar{V}) - X(t,\cdot)$ must be an element of the *polar cone* $N_{\mathscr{K}}(X(t,\cdot))$ of $\mathscr{K}$, which is defined as

$$N_{\mathscr{K}}(X) := \Big\{ Y \in \mathscr{L}^2(\Omega, \mathsf{m})\colon \int_\Omega Y(X' - X) \leqslant 0 \quad \text{for all } X' \in \mathscr{K} \Big\} \tag{1.6}$$

for all $X \in \mathscr{K}$. We observe that $N_{\mathscr{K}}(X)$ coincides with the subdifferential $\partial I_{\mathscr{K}}(X)$. Since $\mathscr{K}$ is a cone, one can choose $X' = 2X$, $X' = 0$ in (1.6) to obtain that

$$Y \in N_{\mathscr{K}}(X) \quad \Longleftrightarrow \quad \int_\Omega YX = 0, \int_\Omega YX' \leqslant 0 \quad \text{for all } X' \in \mathscr{K}. \tag{1.7}$$

One is therefore naturally led to the problem of characterizing the polar cone of the set of monotone transport maps, beyond the basic definition (1.6). It is shown in [7] that if $Y \in \mathscr{L}^2(\Omega, \mathsf{m})$ is an element of the polar cone $N_{\mathscr{K}}(X)$, then Y coincides with the *derivative of a nonnegative function*. We refer the reader to [7] for more details, and to [4,6,9] for similar results.

In this paper, we will give a generalization of this result to the multi-dimensional case. We are interested in the following setting: We assume that a Borel probability measure ϱ on $\mathbb{R}^d$ is given with finite second moments, so that $\int_{\mathbb{R}^d} |x|^2\, \varrho(dx) < \infty$. We consider the closed convex cone of monotone transport maps

$$\mathscr{K}_\varrho := \Big\{ \boldsymbol{f} \in \mathscr{L}^2(\mathbb{R}^d, \varrho)\colon \boldsymbol{f} \text{ is monotone} \Big\}.$$

Here we call any Borel map $\boldsymbol{f}\colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$ monotone if the support of the induced transport plan $\gamma_{\boldsymbol{f}} := (\mathrm{id}, \boldsymbol{f})\#\varrho$, which is a Borel probability measure on the product space $\mathbb{R}^d \times \mathbb{R}^d$, is a monotone set. Recall that $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d$ is monotone if

$$(y_1 - y_2) \cdot (x_1 - x_2) \geqslant 0 \quad \text{for all } (x_i, y_i) \in \Gamma \text{ with } i = 1..2,$$

where $\cdot$ denotes the Euclidean inner product on $\mathbb{R}^d$. Our goal is to find a representation of elements of the polar cone $\mathscr{K}_\varrho^\perp$ (at the zero map), defined as

$$\mathscr{K}_\varrho^\perp := \Big\{ \boldsymbol{g} \in \mathscr{L}^2(\mathbb{R}^d, \varrho)\colon \int_{\mathbb{R}^d} \boldsymbol{g}(x) \cdot \boldsymbol{f}'(x)\, \varrho(dx) \leqslant 0 \quad \text{for all } \boldsymbol{f}' \in \mathscr{K}_\varrho \Big\}.$$

Notice that since ϱ has finite second moments, any smooth monotone function with at most linear growth at infinity (see details below) is an element of $\mathscr{K}_\varrho$. Moreover, whenever $\boldsymbol{g} \in \mathscr{K}_\varrho^\perp$ is given, then the product $\boldsymbol{g}\varrho$ is in fact an $\mathbb{R}^d$-valued finite Borel measure, because of Cauchy-Schwarz inequality. We will show below in Theorem 2.1 that for any $\boldsymbol{g} \in \mathscr{K}_\varrho^\perp$ the measure $\boldsymbol{g}\varrho$ can be written as the divergence of a finite Borel measure taking values in the symmetric, positive semidefinite matrices. In the one-dimensional case, we therefore obtain the derivative of a nonnegative function (measure) as in [7]. Our proof relies on an application of the Hahn-Banach theorem and is inspired by a similar argument in [2] for the construction of Michell trusses.

It is possible to prove a representation of the polar cone $\mathscr{K}_\varrho^\perp$ similar to ours by using a characterization of polar cones from [10] and subharmonic functions; see [6, 9] for instance. Compared to these presentations, our proof is shorter and simpler.

2. The Main Result

We will denote by $x \cdot y$ the Euclidean inner product of $x, y \in \mathbb{R}^k$, and by $|x|$ the induced norm. We write $\mathbb{R}^{l\times l}$ for the space of real matrices. For any $A, B \in \mathbb{R}^{l\times l}$ with components $A = (a_{ij})$ and $B = (b_{ij})$ we define an inner product

$$\langle A, B\rangle := \operatorname{tr}(AB^{\mathrm{T}}) = \sum_{i,j=1}^{l} a_{i,j} b_{i,j}$$

(with B^{T} the transpose of B), which induces the Frobenius norm

$$\|A\| := \sqrt{\operatorname{tr}(AA^{\mathrm{T}})} = \sum_{i,j=1}^{l} a_{i,j}^2.$$

We denote by $\mathcal{S}^l$ the space of symmetric real matrices and by $\mathcal{S}^l_+$ the subset of positive semidefinite symmetric matrices. The space of all positive definite, but not necessarily symmetric matrices will be denoted by $\mathbb{R}^{l\times l}_+$. Recall that

$$A \in \mathbb{R}^{l\times l}_+ \quad\Longleftrightarrow\quad v \cdot (Av) \geqslant 0 \quad \text{for all } v \in \mathbb{R}^l.$$

Equivalently, we have $A \in \mathbb{R}^{l\times l}_+$ if and only if $A^{\mathrm{sym}} := (A + A^{\mathrm{T}})/2 \in \mathcal{S}^l_+$.

Let $\mathscr{C}_*(\mathbb{R}^d; \mathbb{R}^{l\times l})$ be the space of all continuous functions $w\colon \mathbb{R}^d \longrightarrow \mathbb{R}^{l\times l}$ with the property that $\lim_{|x|\to\infty} w(x) \in \mathbb{R}^{l\times l}$ exists. Note that we can write

$$\mathscr{C}_*(\mathbb{R}^d; \mathbb{R}^{l\times l}) = \mathbb{R}^{l\times l} + \mathscr{C}_0(\mathbb{R}^d; \mathbb{R}^{l\times l}),$$

where $\mathscr{C}_0(\mathbb{R}^d; \mathbb{R}^{l\times l})$ is the closure of the space of all compactly supported continuous $\mathbb{R}^{l\times l}$-valued maps, w.r.t. the sup-norm. In an analogous way, we define $\mathscr{C}_*(\mathbb{R}^d; \mathcal{S}^l)$ and $\mathscr{C}_*(\mathbb{R}^d; \mathcal{S}^l_+)$. For any map $u \in \mathscr{C}^1(\mathbb{R}^d; \mathbb{R}^d)$ we denote by

$$e(u(x)) := Du(x)^{\mathrm{sym}} \quad \text{for all } x \in \mathbb{R}^d$$

its deformation tensor, which is an element of $\mathscr{C}(\mathbb{R}^d; \mathcal{S}^d)$. Let

$$\mathscr{C}^1_*(\mathbb{R}^d; \mathbb{R}^d) := \{u \in \mathscr{C}^1(\mathbb{R}^d; \mathbb{R}^d)\colon Du \in \mathscr{C}_*(\mathbb{R}^d; \mathbb{R}^{d\times d})\},$$
$$\mathrm{MON}(\mathbb{R}^d) := \{u \in \mathscr{C}^1_*(\mathbb{R}^d; \mathbb{R}^d)\colon u \text{ is monotone}\},$$

so that $e(u) \in \mathscr{C}_*(\mathbb{R}^d; \mathcal{S}^d_+)$ if $u \in \mathrm{MON}(\mathbb{R}^d)$. The cone $\mathrm{MON}(\mathbb{R}^d)$ contains all linear maps $u(x) := Ax$ for $x \in \mathbb{R}^d$, where $A \in \mathbb{R}^{d\times d}_+$. See [1] for more details.

We will denote by $\mathscr{M}(\mathbb{R}^d; \mathbb{R}^k)$ the space of finite $\mathbb{R}^k$-valued Borel measures. In an analogous way, we define $\mathscr{M}(\mathbb{R}^d; \mathcal{S}^l)$ and $\mathscr{M}(\mathbb{R}^d; \mathcal{S}^l_+)$. If f_i, $i = 1 \dots k$, are the components of $\mathbf{F} \in \mathscr{M}(\mathbb{R}^d; \mathbb{R}^k)$ and $u \in \mathscr{C}_b(\mathbb{R}^d; \mathbb{R}^k)$ we write

$$\int_{\mathbb{R}^d} u(x) \cdot \mathbf{F}(dx) = \sum_{i=1}^{k} u_i(x)\, f_i(dx).$$

We will say that $\mathbf{F}$ has finite first moment if $\sum_{i=1}^k \int_{\mathbb{R}^d} |x|\,|f_i|(dx) < \infty$. If $\mu_{i,j} = \mu_{j,i}$, $i,j = 1\dots l$, are the components of $\mathbf{M} \in \mathscr{M}(\mathbb{R}^d;\mathcal{S}^l)$ and $v \in \mathscr{C}_b(\mathbb{R}^d;\mathcal{S}^l)$, then

$$\int_{\mathbb{R}^d} \langle v(x), \mathbf{M}(dx)\rangle = \sum_{i,j=1}^l v_{i,j}(x)\,\mu_{i,j}(dx).$$

For any $\mathbf{M} = (\mu_{i,j}) \in \mathscr{M}(\mathbb{R}^d;\mathcal{S}^l)$ we have $\mathbf{M} \in \mathscr{M}(\mathbb{R}^d;\mathcal{S}^l_+)$ if and only if

$$\sum_{i,j=1}^l \mu_{i,j} v_i v_j \quad \text{is a positive measure for all } v \in \mathbb{R}^l.$$

We can now state our representation result.

Theorem 2.1 (Stress Tensor)**.** *Assume that there exist a measure $\mathbf{F} \in \mathscr{M}(\mathbb{R}^d;\mathbb{R}^d)$ with finite first moment and a matrix-valued field $\mathbf{H} \in \mathscr{M}(\mathbb{R}^d;\mathcal{S}^d_+)$ with*

$$G(u) := -\int_{\mathbb{R}^d} u(x)\cdot\mathbf{F}(dx) - \int_{\mathbb{R}^d} \langle e(u(x)), \mathbf{H}(dx)\rangle \geqslant 0 \tag{2.1}$$

for all $u \in \mathrm{MON}(\mathbb{R}^d)$. Then there exists $\mathbf{M} \in \mathscr{M}(\mathbb{R}^d;\mathcal{S}^d_+)$ such that

$$G(u) = \int_{\mathbb{R}^d} \langle e(u(x)), \mathbf{M}(dx)\rangle \quad \textit{for all } u \in \mathscr{D}(\mathbb{R}^d;\mathbb{R}^d), \tag{2.2}$$

$$\int_{\mathbb{R}^d} \mathrm{tr}(\mathbf{M}(dx)) \leqslant -\int_{\mathbb{R}^d} x\cdot\mathbf{F}(dx) - \int_{\mathbb{R}^d} \mathrm{tr}(\mathbf{H}(dx)). \tag{2.3}$$

Notice that the integrals in (2.1) are finite for any choice of $u \in \mathscr{C}^1_*(\mathbb{R}^d;\mathbb{R}^d)$, by our assumptions on $\mathbf{F}$ and $\mathbf{H}$. Recall that the trace of a symmetric matrix is equal to the sum of its eigenvalues, which in the case of a positive semidefinite matrix are all nonnegative. Therefore (2.3) controls the size of the measure $\mathbf{M}$.

For $\mathbf{H} \equiv 0$ we obtain the representation announced in the introduction:

$$\int_{\mathbb{R}^d} u(x)\cdot\mathbf{F}(dx) = -\int_{\mathbb{R}^d} \langle Du(x), \mathbf{M}(dx)\rangle$$

for all test functions u. Recall that $\mathbf{M}$ takes values in the symmetric matrices. The more general form of (2.1) is motivated by a variational time discretization for the compressible Euler equations, for which a minimization problem of the form

$$\inf_{\boldsymbol{f}\in\mathscr{K}_\varrho} \left\{ \frac{1}{2}\int_{\mathbb{R}^d} |\boldsymbol{h}(x) - \boldsymbol{f}(x)|^2\,\varrho(dx) + \int_{\mathbb{R}^d} e(x)\det(D\boldsymbol{f}(x)^{\mathrm{sym}})^{1-\gamma}\,dx \right\} \tag{2.4}$$

for suitable $\boldsymbol{h} \in \mathscr{L}^2(\mathbb{R}^d,\varrho)$ and nonnegative $e \in \mathscr{L}^1(\mathbb{R}^d)$ must be solved, with $\gamma > 1$ some constant. Denoting by $\boldsymbol{f} \in \mathscr{K}_\varrho$ the minimizer of (2.4) and letting $\boldsymbol{g} := \boldsymbol{h} - \boldsymbol{f}$, we can write the corresponding first-order optimality condition (formally) as

$$\begin{aligned} &-\int_{\mathbb{R}^d} \boldsymbol{g}(x)\cdot\boldsymbol{f}'(x)\,\varrho(dx) \\ &\qquad -(\gamma-1)\int_{\mathbb{R}^d} e(x)\det(D\boldsymbol{f}(x)^{\mathrm{sym}})^{-\gamma}\ \mathrm{tr}\Big(\mathrm{cof}(D\boldsymbol{f}(x)^{\mathrm{sym}})^{\mathrm{T}} D\boldsymbol{f}'(x)\Big)\,dx \geqslant 0 \end{aligned}$$

for all $\boldsymbol{f}' \in \mathscr{K}_\varrho$. From this, assumption (2.1) follows if we define

$$\mathbf{F} := \boldsymbol{g}\varrho \quad \text{and} \quad \mathbf{H} := (\gamma-1)e\det(D\boldsymbol{f}^{\mathrm{sym}})^{-\gamma}\,\mathrm{cof}(D\boldsymbol{f}^{\mathrm{sym}})^{\mathrm{T}}.$$

One can then check that $\mathbf{F}$ has finite first moments and that $\mathbf{H} \in \mathscr{M}(\mathbb{R}^d;\mathcal{S}^d_+)$. This application will be discussed in more detail in an upcoming publication.

2.1. **Positive Functionals.** In this section, we will discuss a general result about extensions of positive functionals, which is due to Riedl [8]. Let us start with some notation: In the following, we denote by E a normed vector space. We call *positive cone* any subset $C \subset E$ with $C \neq E$ with the following properties:

$$C + C \subset C, \qquad \lambda C \subset C \quad \text{for all } \lambda > 0, \qquad C \cap (-C) = \{0\}. \tag{2.5}$$

The positive cone C induces a partial ordering $\geqslant$ on the space E by

$$y \geqslant x \quad \Longleftrightarrow \quad y - x \in C.$$

A linear map $F\colon L \longrightarrow \mathbb{R}$ defined on a subspace $L \subset E$ is called *positive* if

$$F(x) \geqslant 0 \quad \text{for all } x \in L \cap C. \tag{2.6}$$

A linear map $F\colon E \longrightarrow \mathbb{R}$ is called *functional* if it is continuous.

Proposition 2.2. *Let E be a Banach space, partially ordered by a positive cone C. If some subspace $L \subset E$ contains an interior point of C, then every positive linear map $F_0\colon L \longrightarrow \mathbb{R}$ can be extended to a positive functional $F\colon E \longrightarrow \mathbb{R}$.*

Proof. See Theorem 10.10 of [8]. We include a proof for the reader's convenience.

Step 1. We first observe that $E = L - C$. Indeed if $x_0 \in L$ is an inner point of C, then there exists a $\delta > 0$ with $B_\delta(x_0) \subset C$. Moreover, for all $x \in E$ there exists $\lambda > 0$ (choose $\lambda := 2\|x\|/\delta$, for example) with the property that

$$x/\lambda \subset B_\delta(0) = x_0 - B_\delta(x_0) \subset x_0 - C.$$

Since L is a subspace we obtain, using $\lambda C \subset C$ for all $\lambda > 0$, that

$$E \subset \bigcup_{\lambda > 0} \lambda(x_0 - C) \subset \mathbb{R}x_0 - C \subset L - C.$$

Step 2. Since $E = L - C$, for every $x \in E$ there exist $y_\pm \in L$ and $z_\pm \in C$ such that $\pm x = y_\pm - z_\pm$, which implies that $y_+ \geqslant x \geqslant -y_-$. We now define

$$p(x) := \inf \Big\{ F_0(y) \colon y \in L, y \geqslant x \Big\} \quad \text{for all } x \in E. \tag{2.7}$$

Then $p(x) \leqslant F_0(y_+) < \infty$. On the other hand, for every $y \in L$ with $y \geqslant x$ we have $y \geqslant -y_-$. Since $y + y_- \in L \cap C$, we have $F_0(y + y_-) \geqslant 0$, by positivity of F_0. This implies that $F_0(y) \geqslant -F_0(y_-) > -\infty$. We conclude that $p(x)$ is finite for all $x \in E$. It is easy to check that for all $x_1, x_2 \in E$ and for all $\lambda > 0$ we have

$$p(x_1 + x_2) \leqslant p(x_1) + p(x_2), \qquad p(\lambda x_1) = \lambda p(x_1).$$

For every $x \in L$ and $z \in E$ with $z \geqslant x$, we have $F_0(x) \leqslant p(z)$ (in particular, we may choose $z = x$). Indeed for every $y \in L$ with $y \geqslant z$, we have $y \geqslant x$, thus $y - x \in L \cap C$. Hence $F_0(y - x) \geqslant 0$, by positivity, which yields $F_0(y) = F_0(x) + F_0(y - x) \geqslant F_0(x)$. Taking the inf over all $y \in L$ with $y \geqslant z$, we obtain the estimate.

Step 3. We can now apply the Hahn-Banach theorem and obtain a linear map $F\colon E \longrightarrow \mathbb{R}$ with $F(x) \leqslant p(x)$ for all $x \in E$. In order to show that F is positive, let $x \in C$. Then $0 \geqslant -x$ and $0 \in L$, so we may choose $y = 0$ in the definition of $p(-x)$ (see (2.7)) to obtain $p(-x) \leqslant 0$. Therefore $F(-x) \leqslant p(-x) \leqslant 0$, and so $F(x) \geqslant 0$ for all $x \in C$. To prove that F is an extension of F_0, let $x \in L$. Then we may choose $y = -x$ in (2.7) to obtain $p(-x) \leqslant F_0(-x)$ for all $x \in L$. Then

$$-F(x) = F(-x) \leqslant p(-x) \leqslant F_0(-x) = -F_0(x),$$

hence $F_0(x) \leqslant F(x)$. Applying the same argument to $-x \in L$, we get $F_0(x) \geqslant F(x)$. It follows that $F_0(x) = F(x)$ for all $x \in L$. Therefore F is an extension of F_0.

Step 4. It remains to prove that F is continuous. Let x_0 be the interior point of C from Step 1, for which $B_\delta(x_0) \subset C$. Then $B_\delta(0) \subset \pm(x_0 - C)$. Let $\lambda := F(x_0) \geqslant 0$ (recall that $F(x) \geqslant 0$ for all $x \in C$). Then for all $x \in B_\delta(0)$ we have $x_0 - x \in C$, thus $F(x_0 - x) \geqslant 0$. It follows that $F(x_0) \geqslant F(x)$. Similarly, we obtain $F(x) \geqslant -F(x_0)$. Then either F vanishes (if $\lambda = 0$), or the preimage of the nonempty interval $(-\lambda, \lambda)$ contains a neighborhood of 0, and so F (being linear) is continuous. □

2.2. **Proof of Theorem 2.1.** We apply Proposition 2.2 with

$$E := \mathscr{C}_*(\mathbb{R}^d; \mathcal{S}^d), \quad C := \mathscr{C}_*(\mathbb{R}^d, \mathcal{S}^d_+), \quad L := \{e(u) \colon u \in \mathscr{C}^1_*(\mathbb{R}^d; \mathbb{R}^d)\}.$$

Clearly C satisfies conditions (2.5). The identity map id is an element of $\mathrm{MON}(\mathbb{R}^d)$, with constant deformation tensor $e(\mathrm{id})$ equal to the identity matrix $\mathbf{1} \in \mathcal{S}^d_+$. Since the eigenvalues of a symmetric matrix depend continuously on the matrix entries, we have that $e(\mathrm{id}) = \mathbf{1}$ is an interior point of C: For all $\|v - \mathrm{id}\|_E$ sufficiently small, the eigenvalues of $v(x)$ are bigger than $1/2$ for all $x \in \mathbb{R}^d$ and $v \in E$.

On the subspace $L \subset E$, we define the functional F_0 as

$$F_0(v) := -\int_{\mathbb{R}^d} u(x) \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \langle v(x), \mathbf{H}(dx) \rangle \quad \text{where } v = e(u).$$

Note that F_0 is well-defined: If there exists another map $\tilde{u} \in \mathscr{C}^1_*(\mathbb{R}^d; \mathbb{R}^d)$ such that $e(\tilde{u}(x)) = v(x)$ for all $x \in \mathbb{R}^d$, then we have $e(u - \tilde{u}) \equiv 0$, by linearity. Consequently, there exist an antisymmetric matrix $B \in \mathbb{R}^{d \times d}$ and $c \in \mathbb{R}^d$ such that

$$\bar{u}(x) := u(x) - \tilde{u}(x) = Bx + c \quad \text{for all } x \in \mathbb{R}^d.$$

Indeed assume that $e(\bar{u}(x)) = 0$ and define

$$W\bar{u}(x) := \frac{D\bar{u}(x) - D\bar{u}(x)^{\mathrm{T}}}{2} \quad \text{for all } x \in \mathbb{R}^d.$$

Then $\partial_k (W\bar{u})_{i,j} \equiv 0$ for all indices i, j, k. Since $D\bar{u} = e(\bar{u}) + W\bar{u}$ it follows that $D\bar{u}$ is a constant matrix with vanishing symmetric part, so $\bar{u}$ is a rigid deformation. We now observe that both $\pm\bar{u} \in \mathrm{MON}(\mathbb{R}^d)$, which implies $F_0(e(\bar{u})) = 0$ because of (2.1). As F_0 is linear, we conclude that F_0 is well-defined. Similarly, one can check that $F_0(v) \geqslant 0$ for all $v \in L \cap C$, so the linear map $F_0 \colon L \longrightarrow \mathbb{R}$ is positive.

Applying Proposition 2.2, we obtain that F_0 can be extended to a continuous linear map $F \colon E \longrightarrow \mathbb{R}$. Notice that $\mathscr{C}_*(\mathbb{R}^d; \mathbb{R})$ is a separable and *closed subalgebra* of the space $\mathscr{C}_b(\mathbb{R}^d; \mathbb{R})$ of bounded continuous $\mathcal{S}^d$-functions. As is well-known, to any closed subalgebra of a space of bounded continuous functions, there corresponds a compactification of the domain. In our case, we obtain the one-point (also called Alexandroff) compactification of $\mathbb{R}^d$, which we will denote by $\beta\mathbb{R}^d$. Then $\mathscr{C}_*(\mathbb{R}^d; \mathcal{S}^d)$ is isomorphic to $\mathscr{C}(\beta\mathbb{R}^d; \mathcal{S}^d)$. We refer the reader to [5] Section 4.8 for more details. By the Riesz representation theorem, there therefore exists a finite Radon measure $\mathbf{M} \in \mathscr{M}(\beta\mathbb{R}^d; \mathcal{S}^d)$ that represents the functional F in the sense that

$$F(v) = \int_{\beta\mathbb{R}^d} \langle v(x), \mathbf{M}(dx) \rangle \quad \text{for all } v \in \mathscr{C}_*(\mathbb{R}^d; \mathcal{S}^d).$$

Since $F(v) \geqslant 0$ for all $v \in \mathscr{C}_*(\mathbb{R}^d; \mathcal{S}^d_+)$ we obtain that $\mathbf{M}$ takes in fact values in $\mathcal{S}^d_+$. Moreover, as F is an extension of F_0, the following identity holds:

$$F_0(v) = -\int_{\mathbb{R}^d} u(x) \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \langle v(x), \mathbf{H}(dx) \rangle = \int_{\beta\mathbb{R}^d} \langle v(x), \mathbf{M}(dx) \rangle$$

for any $v = e(u)$ and $u \in \mathscr{C}^1_*(\mathbb{R}^d; \mathbb{R}^d)$; see (2.2). In particular, we may choose $u = \mathrm{id}$ (with $e(\mathrm{id}) = \mathbf{1}$) to obtain the control (recall that $\mathbf{M}$ is $\mathcal{S}^d_+$-valued)

$$\int_{\beta\mathbb{R}^d} \mathrm{tr}(\mathbf{M}(dx)) = -\int_{\mathbb{R}^d} x \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \mathrm{tr}(\mathbf{H}(dx)).$$

Restricting the representation from $\beta\mathbb{R}^d$ to $\mathbb{R}^d$, we obtain the result.

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