THE POLAR CONE OF THE SET OF MONOTONE MAPS

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Abstract. We prove that every element of the polar cone to the closed convex cone of monotone transport maps can be represented as the divergence of a measure field taking values in the positive definite matrices.

Contents

1. Introduction 1
2. The Main Result 3
   2.1. Positive Functionals 5
   2.2. Proof of Theorem 2.1 6
References 7

1. Introduction

The one-dimensional pressureless gas dynamics equations
\[
\begin{align*}
\partial_t \varrho + \nabla \cdot (\varrho u) &= 0 \\
\partial_t (\varrho u) + \nabla \cdot (\varrho u \otimes u) &= 0
\end{align*}
\]
(1.1)
in \([0, \infty) \times \mathbb{R}\) has recently been shown equivalent (in the regime of sticky particles) to a first-order differential inclusion on the space of monotone transport maps from the reference measure space \(([0, 1], \mathcal{L}^1|[0, 1]) =: (\Omega, m)\) (where \(\mathcal{L}^1\) is the one-dimensional Lebesgue measure) to \(\mathbb{R}\); see [7]. More precisely, to every density/velocity \((\varrho, u)\) solving (1.1) one can associate a unique map \(X \in \mathcal{L}^2(\Omega, m)\) such that
\[
\varrho(t, \cdot) = X(t, \cdot) m \quad \text{for all } t \in [0, \infty).
\]
(1.2)
Here \# indicates the push-forward of measures. Then \(X\) satisfies
\[
\dot{X} + \partial I_{\mathcal{X}}(X) \ni \bar{V} \quad \text{for all } t \in [0, \infty),
\]
(1.3)
where \(\mathcal{X}\) denotes the closed convex cone of all transport maps \(X \in \mathcal{L}^2(\Omega, m)\) that are monotone, and where \(\partial I_{\mathcal{X}}\) is the subdifferential of the indicator function of \(\mathcal{X}\). If \(X\) satisfies (1.3) and is related to \(\varrho\) through (1.2), then the Eulerian velocity \(u\) can be recovered from the Lagrangian velocity \(V := \dot{X}\) through
\[
V(t, \cdot) = u(t, X(t, \cdot)) \quad \text{for all } t \in [0, \infty).
\]
(1.4)
Assuming finite kinetic energy, it is natural to require that
\[
V(t, \cdot) \in \mathcal{L}^2(\Omega, m), \quad u(t, \cdot) \in \mathcal{L}^2(\mathbb{R}, \varrho(t, \cdot)).
\]
The relation (1.4) in particular determines the initial Lagrangian velocity $\dot{V}$ in (1.3) in terms of the initial data $(\rho, u)(0, \cdot) = (\bar{\rho}, \bar{u})$ of the system (1.1).

It is shown in [7] that the solution of (1.3) can be written explicitly as

$$X(t, \cdot) = P_{\mathcal{K}}(\bar{X} + tv) \quad \text{for all } t \in [0, \infty),$$

with $\bar{X} := X(0, \cdot) \in \mathcal{K}$ given by (1.2). Here $P_\mathcal{K}$ denotes the metric projection onto the cone $\mathcal{K}$. The connection between (1.1) and (1.3) makes it possible to apply classical results from the theory of first-order differential inclusions in Hilbert spaces to study the pressureless gas dynamics equations, which form a system of hyperbolic conservation laws. We refer the reader to [3] for further information.

It is known that if $X$ satisfies (1.5), then the difference $(\bar{X} + tv) - X(t, \cdot)$ must be an element of the polar cone $N_\mathcal{K}(X(t, \cdot))$ of $\mathcal{K}$, which is defined as

$$N_\mathcal{K}(X) := \left\{ Y \in \mathcal{L}^2(\Omega, m) : \int_{\Omega} Y(X' - X) \leq 0 \quad \text{for all } X' \in \mathcal{K} \right\}$$

for all $X \in \mathcal{K}$. We observe that $N_\mathcal{K}(X)$ coincides with the subdifferential $\partial I_\mathcal{K}(X)$. Since $\mathcal{K}$ is a cone, one can choose $X' = 2X$, $X' = 0$ in (1.6) to obtain that

$$Y \in N_\mathcal{K}(X) \iff \int_{\Omega} YX = 0, \int_{\Omega} YX' \leq 0 \quad \text{for all } X' \in \mathcal{K}. \quad (1.7)$$

One is therefore naturally led to the problem of characterizing the polar cone of the set of monotone transport maps, beyond the basic definition (1.6). It is shown in [7] that if $Y \in \mathcal{L}^2(\Omega, m)$ is an element of the polar cone $N_\mathcal{K}(X)$, then $Y$ coincides with the derivative of a nonnegative function. We refer the reader to [7] for more details, and to [4,6,9] for similar results.

In this paper, we will give a generalization of this result to the multi-dimensional case. We are interested in the following setting: We assume that a Borel probability measure $\rho$ on $\mathbb{R}^d$ is given with finite second moments, so that $\int_{\mathbb{R}^d} |x|^2 \rho(dx) < \infty$. We consider the closed convex cone of monotone transport maps

$$\mathcal{K}_\rho := \left\{ f \in \mathcal{L}^2(\mathbb{R}^d, \rho) : f \text{ is monotone} \right\}.$$

Here we call any Borel map $f : \mathbb{R}^d \to \mathbb{R}^d$ monotone if the support of the induced transport plan $\gamma_f := (\text{id}, f)\# \rho$, which is a Borel probability measure on the product space $\mathbb{R}^d \times \mathbb{R}^d$, is a monotone set. Recall that $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d$ is monotone if

$$(y_1 - y_2) \cdot (x_1 - x_2) \geq 0 \quad \text{for all } (x_i, y_i) \in \Gamma \text{ with } i = 1, 2,$$

where $\cdot$ denotes the Euclidean inner product on $\mathbb{R}^d$. Our goal is to find a representation of elements of the polar cone $\mathcal{K}_\rho^\perp$ (at the zero map), defined as

$$\mathcal{K}_\rho^\perp := \left\{ g \in \mathcal{L}^2(\mathbb{R}^d, \rho) : \int_{\mathbb{R}^d} g(x) \cdot \dot{f}'(x) \rho(dx) \leq 0 \quad \text{for all } \dot{f}' \in \mathcal{K}_\rho^\perp \right\}.$$

Notice that since $\rho$ has finite second moments, any smooth monotone function with at most linear growth at infinity (see details below) is an element of $\mathcal{K}_\rho$. Moreover, whenever $g \in \mathcal{K}_\rho^\perp$ is given, then the product $g\rho$ is in fact an $\mathbb{R}^d$-valued finite Borel measure, because of Cauchy-Schwarz inequality. We will show below in Theorem 2.1 that for any $g \in \mathcal{K}_\rho^\perp$ the measure $g\rho$ can be written as the divergence of a finite Borel measure taking values in the symmetric, positive semidefinite matrices. In the one-dimensional case, we therefore obtain the derivative of a nonnegative function (measure) as in [7]. Our proof relies on an application of the Hahn-Banach theorem and is inspired by a similar argument in [2] for the construction of Michell trusses.
It is possible to prove a representation of the polar cone $\mathcal{K}_g^-$ similar to ours by using a characterization of polar cones from [10] and subharmonic functions; see [6,9] for instance. Compared to these presentations, our proof is shorter and simpler.

2. The Main Result

We will denote by $x \cdot y$ the Euclidean inner product of $x, y \in \mathbb{R}^k$, and by $|x|$ the induced norm. We write $\mathbb{R}^{l \times l}$ for the space of real matrices with components $A = (a_{ij})$ and $B = (b_{ij})$ we define an inner product

$$\langle A, B \rangle := \text{tr}(AB^T) = \sum_{i,j=1}^l a_{i,j}b_{i,j}$$

(with $B^T$ the transpose of $B$), which induces the Frobenius norm

$$\|A\| := \sqrt{\text{tr}(AA^T)} = \sum_{i,j=1}^l a_{i,j}^2.$$  

We denote by $\mathcal{S}^l$ the space of symmetric real matrices and by $\mathcal{S}^l_+$ the subset of positive semidefinite symmetric matrices. The space of all positive definite, but not necessarily symmetric matrices will be denoted by $\mathbb{R}^{l \times l}_+$. Recall that

$$A \in \mathbb{R}^{l \times l}_+ \iff v \cdot (Av) \geq 0 \text{ for all } v \in \mathbb{R}^l.$$  

Equivalently, we have $A \in \mathbb{R}^{l \times l}_+$ if and only if $A_{\text{sym}} := (A + A^T)/2 \in \mathcal{S}^l_+$.

Let $\mathcal{C}_s(\mathbb{R}^d; \mathbb{R}^{l \times l})$ be the space of all continuous functions $w: \mathbb{R}^d \to \mathbb{R}^{l \times l}$ with the property that $\lim_{|x| \to \infty} w(x) \in \mathbb{R}^{l \times l}$ exists. Note that we can write

$$\mathcal{C}_s(\mathbb{R}^d; \mathbb{R}^{l \times l}) = \mathbb{R}^{l \times l} + \mathcal{C}_0(\mathbb{R}^d, \mathbb{R}^{l \times l}),$$

where $\mathcal{C}_0(\mathbb{R}^d, \mathbb{R}^{l \times l})$ is the closure of the space of all compactly supported continuous $\mathbb{R}^{l \times l}$-valued maps, w.r.t. the sup-norm. In an analogous way, we define $\mathcal{C}_s(\mathbb{R}^d, \mathcal{S}^l)$ and $\mathcal{C}_s(\mathbb{R}^d, \mathcal{S}^l_+)$. For any map $u \in \mathcal{C}_1(\mathbb{R}^d; \mathbb{R}^d)$ we denote by

$$e(u(x)) := Du(x)^{\text{sym}} \text{ for all } x \in \mathbb{R}^d$$

its deformation tensor, which is an element of $\mathcal{C}(\mathbb{R}^d; \mathcal{S}^d)$. Let

$$\mathcal{C}_s^1(\mathbb{R}^d, \mathbb{R}^d) := \{ u \in \mathcal{C}_s^1(\mathbb{R}^d, \mathbb{R}^d): Du \in \mathcal{C}_s(\mathbb{R}^d, \mathbb{R}^{d \times d}) \},$$

$$\text{MON}(\mathbb{R}^d) := \{ u \in \mathcal{C}_s^1(\mathbb{R}^d, \mathbb{R}^d): u \text{ is monotone} \},$$

so that $e(u) \in \mathcal{C}_s(\mathbb{R}^d; \mathcal{S}^d_+)$ if $u \in \text{MON}(\mathbb{R}^d)$. The cone $\text{MON}(\mathbb{R}^d)$ contains all linear maps $u(x) := Ax$ for $x \in \mathbb{R}^d$, where $A \in \mathbb{R}^{d \times d}$. See [1] for more details.

We will denote by $\mathcal{M}(\mathbb{R}^d; \mathbb{R}^k)$ the space of finite $\mathbb{R}^k$-valued Borel measures. In an analogous way, we define $\mathcal{M}(\mathbb{R}^d; \mathcal{S}^l)$ and $\mathcal{M}(\mathbb{R}^d; \mathcal{S}^l_+)$. If $f_i, i = 1 \ldots k$, are the components of $F \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^k)$ and $u \in \mathcal{C}_b(\mathbb{R}^d; \mathbb{R}^k)$ we write

$$\int_{\mathbb{R}^d} u(x) \cdot F(dx) = \sum_{i=1}^k u_i(x) f_i(dx).$$
We will say that $F$ has finite first moment if $\sum_{i=1}^{n} \int_{\mathbb{R}^d} |f_i|(dx) < \infty$. If $\mu_{i,j} = \mu_{j,i}$, $i,j = 1 \ldots l$, are the components of $M \in \mathcal{M}(\mathbb{R}^d; \mathbb{S}^l)$ and $v \in \mathcal{C}_b(\mathbb{R}^d; \mathbb{S}^l)$, then

$$\int_{\mathbb{R}^d} \langle v(x), M(dx) \rangle = \sum_{i,j=1}^{l} v_{i,j}(x) \mu_{i,j}(dx).$$

For any $M = (\mu_{i,j}) \in \mathcal{M}(\mathbb{R}^d; \mathbb{S}^l)$ we have $M \in \mathcal{M}(\mathbb{R}^d; \mathbb{S}^l)$ if and only if

$$\sum_{i,j=1}^{l} \mu_{i,j} v_i v_j$$

is a positive measure for all $v \in \mathbb{R}^l$.

We can now state our representation result.

**Theorem 2.1 (Stress Tensor).** Assume that there exist a measure $F \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ with finite first moment and a matrix-valued field $H \in \mathcal{M}(\mathbb{R}^d; \mathbb{S}_l)$ with

$$G(u) := -\int_{\mathbb{R}^d} u(x) \cdot F(dx) - \int_{\mathbb{R}^d} \langle e(u(x)), H(dx) \rangle \geq 0$$

for all $u \in \text{MON}(\mathbb{R}^d)$. Then there exists $M \in \mathcal{M}(\mathbb{R}^d; \mathbb{S}_l^+)$ such that

$$G(u) = \int_{\mathbb{R}^d} \langle e(u(x)), M(dx) \rangle \text{ for all } u \in \mathcal{D}(\mathbb{R}^d; \mathbb{R}^d),$$

$$\int_{\mathbb{R}^d} \text{tr}(M(dx)) \leq -\int_{\mathbb{R}^d} x \cdot F(dx) - \int_{\mathbb{R}^d} \text{tr}(H(dx)).$$

Notice that the integrals in (2.1) are finite for any choice of $u \in \mathcal{C}_1^1(\mathbb{R}^d; \mathbb{R}^d)$, by our assumptions on $F$ and $H$. Recall that the trace of a symmetric matrix is equal to the sum of its eigenvalues, which in the case of a positive semidefinite matrix are all nonnegative. Therefore (2.3) controls the size of the measure $M$.

For $H \equiv 0$ we obtain the representation announced in the introduction:

$$\int_{\mathbb{R}^d} u(x) \cdot F(dx) = -\int_{\mathbb{R}^d} \langle Du(x), M(dx) \rangle$$

for all test functions $u$. Recall that $M$ takes values in the symmetric matrices. The more general form of (2.1) is motivated by a variational time discretization for the compressible Euler equations, for which a minimization problem of the form

$$\inf_{f \in \mathcal{X}_g} \left\{ \frac{1}{2} \int_{\mathbb{R}^d} |h(x) - f(x)|^2 \rho(dx) + \int_{\mathbb{R}^d} e(x) \det(Df(x)^{sym})^{1-\gamma} dx \right\},$$

for suitable $h \in L^2(\mathbb{R}^d, \rho)$ and nonnegative $e \in L^1(\mathbb{R}^d)$ must be solved, with $\gamma > 1$ some constant. Denoting by $f \in \mathcal{X}_g$ the minimizer of (2.4) and letting $g := h - f$, we can write the corresponding first-order optimality condition (formally) as

$$-\int_{\mathbb{R}^d} g(x) \cdot f'(x) \rho(dx)$$

$$- (\gamma - 1) \int_{\mathbb{R}^d} e(x) \det(Df(x)^{sym})^{-\gamma} \text{ tr } \left( \text{cof}(Df(x)^{sym})^T Df'(x) \right) \rho(dx) \geq 0$$

for all $f' \in \mathcal{X}_g$. From this, assumption (2.1) follows if we define

$$F := g \rho \text{ and } H := (\gamma - 1) e \det(Df^{sym})^{-\gamma} \text{cof}(Df^{sym})^T.$$

One can then check that $F$ has finite first moments and that $H \in \mathcal{M}(\mathbb{R}^d; \mathbb{S}^d_+)$. This application will be discussed in more detail in an upcoming publication.
2.1. Positive Functionals. In this section, we will discuss a general result about extensions of positive functionals, which is due to Riedl [8]. Let us start with some notation: In the following, we denote by $E$ a normed vector space. We call positive cone any subset $C \subset E$ with $C \neq E$ with the following properties:

$$C + C \subset C, \quad \lambda C \subset C \quad \text{for all } \lambda > 0, \quad C \cap (-C) = \{0\}.$$ \hfill (2.5)

The positive cone $C$ induces a partial ordering $\geq$ on the space $E$ by

$$y \geq x \iff y - x \in C.$$ 

A linear map $F: L \rightarrow \mathbb{R}$ defined on a subspace $L \subset E$ is called positive if

$$F(x) \geq 0 \quad \text{for all } x \in L \cap C. \hfill (2.6)$$

A linear map $F: E \rightarrow \mathbb{R}$ is called functional if it is continuous.

**Proposition 2.2.** Let $E$ be a Banach space, partially ordered by a positive cone $C$. If some subspace $L \subset E$ contains an interior point of $C$, then every positive linear map $F_0: L \rightarrow \mathbb{R}$ can be extended to a positive functional $F: E \rightarrow \mathbb{R}$.

**Proof.** See Theorem 10.10 of [8]. We include a proof for the reader’s convenience.

**Step 1.** We first observe that $E = L - C$. Indeed if $x_0 \in L$ is an inner point of $C$, then there exists a $\delta > 0$ with $B_\delta(x_0) \subset C$. Moreover, for all $x \in E$ there exists $\lambda > 0$ (choose $\lambda := 2\|x\|/\delta$, for example) with the property that

$$x/\lambda \subset B_\delta(0) = x_0 - B_\delta(x_0) \subset x_0 - C.$$ 

Since $L$ is a subspace we obtain, using $\lambda C \subset C$ for all $\lambda > 0$, that

$$E \subset \bigcup_{\lambda > 0} \lambda(x_0 - C) \subset \mathbb{R}x_0 - C \subset L - C.$$ 

**Step 2.** Since $E = L - C$, for every $x \in E$ there exist $y_\pm \in L$ and $z_\pm \in C$ such that $\pm x = y_\pm - z_\pm$, which implies that $y_+ \geq x \geq -y_-$. We now define

$$p(x) := \inf \left\{ F_0(y) : y \in L, y \geq x \right\} \quad \text{for all } x \in E. \hfill (2.7)$$

Then $p(x) \leq F_0(y_+) < \infty$. On the other hand, for every $y \in L$ with $y \geq x$ we have $y \geq -y_-$. Since $y + y_- \in L \cap C$, we have $F_0(y + y_-) \geq 0$, by positivity of $F_0$. This implies that $F_0(y) \geq -F_0(y_-) > -\infty$. We conclude that $p(x)$ is finite for all $x \in E$.

It is easy to check that for all $x_1, x_2 \in E$ and for all $\lambda > 0$ we have

$$p(x_1 + x_2) \leq p(x_1) + p(x_2), \quad p(\lambda x_1) = \lambda p(x_1).$$

For every $x \in L$ and $z \in E$ with $z \geq x$, we have $F_0(x) \leq p(z)$ (in particular, we may choose $z = x$). Indeed for every $y \in E$ with $y \geq z$, we have $y \geq x$, thus $y - x \in L \cap C$. Hence $F_0(y - x) \geq 0$, by positivity, which yields $F_0(y) = F_0(x) + F_0(y - x) \geq F_0(x)$. Taking the inf over all $y \in L$ with $y \geq z$, we obtain the estimate.

**Step 3.** We can now apply the Hahn-Banach theorem and obtain a linear map $F: E \rightarrow \mathbb{R}$ with $F(x) \leq p(x)$ for all $x \in E$. In order to show that $F$ is positive, let $x \in C$. Then $0 \geq -x$ and $0 \in L$, so we may choose $y = 0$ in the definition of $p(-x)$ (see (2.7)) to obtain $p(-x) \leq 0$. Therefore $F(-x) \leq p(-x) \leq 0$, and so $F(x) \geq 0$ for all $x \in C$. To prove that $F$ is an extension of $F_0$, let $x \in L$. Then we may choose $y = -x$ in (2.7) to obtain $p(-x) \leq F_0(-x)$ for all $x \in L$. Then

$$-F(x) = F(-x) \leq p(-x) \leq F_0(-x) = -F_0(x),$$
hence $F_0(x) \leq F(x)$. Applying the same argument to $-x \in L$, we get $F_0(x) \geq F(x)$. It follows that $F_0(x) = F(x)$ for all $x \in L$. Therefore $F$ is an extension of $F_0$.

**Step 4.** It remains to prove that $F$ is continuous. Let $x_0$ be the interior point of $C$ from Step 1, for which $B_\delta(x_0) \subset C$. Then $B_\delta(0) \subset \pm(x_0-C)$. Let $\lambda := F(x_0) \geq 0$ (recall that $F(x) \geq 0$ for all $x \in C$). Then for all $x \in B_\delta(0)$ we have $x_0-x \in C$, thus $F(x_0-x) \geq 0$. It follows that $F(x_0) \geq F(x)$. Similarly, we obtain $F(x) \geq -F(x_0)$. Then either $F$ vanishes (if $\lambda = 0$), or the preimage of the nonempty interval $(-\lambda, \lambda)$ contains a neighborhood of 0, and so $F$ (being linear) is continuous. \qed

2.2. Proof of Theorem 2.1. We apply Proposition 2.2 with

$$E := \mathcal{C}_b(\mathbb{R}^d; S^d), \quad C := \mathcal{C}_b(\mathbb{R}^d, S^d_+), \quad L := \{e(u): u \in \mathcal{C}_b(\mathbb{R}^d; \mathbb{R}^d)\}.$$ 

Clearly $C$ satisfies conditions (2.5). The identity map id is an element of $\text{MON}(\mathbb{R}^d)$, with constant deformation tensor $e(\text{id})$ equal to the identity matrix $1 \in S^d_+$. Since the eigenvalues of a symmetric matrix depend continuously on the matrix entries, we have that $e(\text{id}) = 1$ is an interior point of $C$: For all $\|v-\text{id}\|_E$ sufficiently small, the eigenvalues of $v(x)$ are bigger than 1/2 for all $x \in \mathbb{R}^d$ and $v \in E$.

On the subspace $L \subset E$, we define the functional $F_0$ as

$$F_0(v) := -\int_{\mathbb{R}^d} u(x) \cdot \mathbf{F}(dx) - \int_{\mathbb{R}^d} \langle v(x), \mathbf{H}(dx) \rangle \quad \text{where} \quad v = e(u).$$

Note that $F_0$ is well-defined: If there exists another map $\tilde{u} \in \mathcal{C}_b^1(\mathbb{R}^d; \mathbb{R}^d)$ such that $e(\tilde{u}(x)) = v(x)$ for all $x \in \mathbb{R}^d$, then we have $e(u-\tilde{u}) \equiv 0$, by linearity. Consequently, there exist an antisymmetric matrix $B \in \mathbb{R}^{d \times d}$ and $c \in \mathbb{R}^d$ such that

$$\tilde{u}(x) := u(x) - \tilde{u}(x) = Bx + c \quad \text{for all} \quad x \in \mathbb{R}^d.$$ 

Indeed assume that $e(\tilde{u}(x)) = 0$ and define

$$W\tilde{u}(x) := \frac{D\tilde{u}(x) - D\tilde{u}(x)^T}{2} \quad \text{for all} \quad x \in \mathbb{R}^d.$$ 

Then $\partial_k(W\tilde{u})_{i,j} \equiv 0$ for all indices $i, j, k$. Since $D\tilde{u} = e(\tilde{u}) + W\tilde{u}$ it follows that $D\tilde{u}$ is a constant matrix with vanishing symmetric part, so $\tilde{u}$ is a rigid deformation. We now observe that both $\pm \tilde{u} \in \text{MON}(\mathbb{R}^d)$, which implies $F_0(e(\tilde{u})) = 0$ because of (2.1). As $F_0$ is linear, we conclude that $F_0$ is well-defined. Similarly, one can check that $F_0(v) \geq 0$ for all $v \in L \cap C$, so the linear map $F_0: L \rightarrow \mathbb{R}$ is positive.

Applying Proposition 2.2, we obtain that $F_0$ can be extended to a continuous linear map $F: E \rightarrow \mathbb{R}$. Notice that $\mathcal{C}_b(\mathbb{R}^d; \mathbb{R})$ is a separable and closed subalgebra of the space $\mathcal{C}_b(\mathbb{R}^d; \mathbb{R})$ of bounded continuous $S^d$-functions. As is well-known, to any closed subalgebra of a space of bounded continuous functions, there corresponds a compactification of the domain. In our case, we obtain the one-point (also called Alexandroff) compactification of $\mathbb{R}^d$, which we will denote by $\beta\mathbb{R}^d$. Then $\mathcal{C}_b(\mathbb{R}^d; S^d)$ is isomorphic to $\mathcal{C}(\beta\mathbb{R}^d; S^d)$. We refer the reader to [5] Section 4.8 for more details. By the Riesz representation theorem, there therefore exists a finite Radon measure $M \in \mathcal{M}(\beta\mathbb{R}^d; S^d)$ that represents the functional $F$ in the sense that

$$F(v) = \int_{\beta\mathbb{R}^d} \langle v(x), M(dx) \rangle \quad \text{for all} \quad v \in \mathcal{C}_b(\mathbb{R}^d; S^d).$$
Since $F(v) \geq 0$ for all $v \in \mathcal{C}^*_r(\mathbb{R}^d; S^d_+)$ we obtain that $M$ takes in fact values in $S^d_+$. Moreover, as $F$ is an extension of $F_0$, the following identity holds:

$$F_0(v) = -\int_{\mathbb{R}^d} u(x) \cdot F(dx) - \int_{\mathbb{R}^d} \langle v(x), H(dx) \rangle = \int_{\mathbb{R}^d} \langle v(x), M(dx) \rangle$$

for any $v = e(u)$ and $u \in \mathcal{C}^1_r(\mathbb{R}^d; \mathbb{R}^d)$; see (2.2). In particular, we may choose $u = \text{id}$ (with $e(\text{id}) = 1$) to obtain the control (recall that $M$ is $S^d_+$-valued)

$$\int_{\mathbb{R}^d} \text{tr}(M(dx)) = -\int_{\mathbb{R}^d} x \cdot F(dx) - \int_{\mathbb{R}^d} \text{tr}(H(dx)).$$

Restricting the representation from $\beta\mathbb{R}^d$ to $\mathbb{R}^d$, we obtain the result.

References


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