A NEW CONVERGENCE PROOF FOR FINITE VOLUME SCHEMES USING THE KINETIC FORMULATION OF CONSERVATION LAWS

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Abstract. We give a new convergence proof for finite volume schemes approximating scalar conservation laws. The main ingredients of the proof are the kinetic formulation of scalar conservation laws, a discrete entropy inequality and the velocity averaging technique.

1. Introduction

We consider the Cauchy problem for nonlinear hyperbolic scalar conservation laws in several space dimensions

$$\frac{\partial \bar{u}}{\partial t} + \nabla \cdot f(\bar{u}) = 0 \qquad \text{on the slab } \mathbf{\Pi} := [0,T) \times \mathbf{R}^{d-1} \tag{1.1}$$

for compactly supported initial data $\bar{u}(0,\cdot) =: \bar{u}^0 \in L^1 \cap L^\infty(\mathbf{R}^{d-1})$. We assume the flux function f in $C^{1,1}_{loc}(\mathbf{R})$ and $f(0) = 0$. As is well known, solutions of nonlinear conservation laws may become discontinuous in finite time, so weak solutions must be considered, i.e. functions $\bar{u} \in L^1 \cap L^\infty(\mathbf{\Pi})$ such that

$$\int_{\mathbf{\Pi}} \left\{ \bar{u}(x) \frac{\partial \phi}{\partial t}(x) + f(\bar{u}(x)) \cdot \nabla \phi(x) \right\} dx + \int_{\mathbf{R}^{d-1}} \bar{u}^0(\bar{x}) \phi(0,\bar{x}) \, d\bar{x} = 0 \tag{1.2}$$

for all $\phi \in \mathcal{D}(\mathbf{\Pi})$. As usual, we require an entropy condition (cf. Lax [La'73]). For any entropy $U \in C^2(\mathbf{R})$ we define the entropy flux

$$F(v) := \int_0^v U'(s) f'(s) \, ds. \tag{1.3}$$

Then the entropy condition reads as follows: For all convex U and $\phi \in \mathcal{D}(\mathbf{\Pi})$, $\phi \geq 0$

$$\int_{\mathbf{\Pi}} \left\{ U(\bar{u}(x)) \frac{\partial \phi}{\partial t}(x) + F(\bar{u}(x)) \cdot \nabla \phi(x) \right\} dx + \int_{\mathbf{R}^{d-1}} U(\bar{u}^0(\bar{x})) \phi(0,\bar{x}) \, d\bar{x} \geq 0. \tag{1.4}$$

A function $\bar{u} \in L^1 \cap L^\infty(\mathbf{\Pi})$ such that (1.2) and (1.4) hold for all convex entropies U will be called a weak entropy solution of the Cauchy problem (1.1).

We are concerned with the convergence of approximations of $\bar{u}$ by finite volume schemes. This question has a history going back to the 1950s. Let us point out two modern developments: The first is Kuznetsov's [Kz'76] approximation theory, that was generalized by Vila [Vi'94] to first-order finite volume methods on unstructured grids and by Cockburn, Coquel and LeFloch [CCL'94] to higher-order schemes.

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Further generalizations can be found in Cockburn and Gremaud [CG'96] and Noelle [No'96]. The second approach is based on a uniqueness result for measure-valued solutions due to DiPerna [Di'85], which was first applied to the analysis of numerical schemes by Szepessy [Sz'89] and Coquel and LeFloch [CL'91, CL'93]. Cockburn, Coquel and LeFloch [CCL'95] and Kröner and Rokyta [KR'94] applied this theory to first-order finite volume schemes, and Kröner, Noelle and Rokyta [KNR'95] to higher-order schemes. Noelle [No'95] extended these results to irregular grids, where cells may become flat as $h \to 0$, and to general E-fluxes, which include Godunov's flux. Both Kuznetsov's and DiPerna's approach rely on Kruzkov's existence and uniqueness result [Kr'70].
In this paper, we give a new convergence proof for finite volume schemes on irregular grids. Our approach combines several results from the field of conservation laws, namely the kinetic formulation for scalar conservation laws, a discrete entropy inequality and the velocity averaging technique.
The kinetic formulation was introduced by Lions, Perthame and Tadmor [LPT'94]. They show that there is a one-to-one correspondence between the entropy solutions of a scalar conservation law and solutions of a linear transport equation for which a certain nonlinear constraint holds true. More precisely, one considers functions $\bar\rho$ depending on space-time and an additional $v \in \mathbf{R}$ that solve the following equation

$$\begin{aligned}(1.5)\qquad &\int_{\mathbf{\Pi}\times\mathbf{R}} \bar\rho(x,v)\left\{\frac{\partial\Phi}{\partial t}(x,v) + f'(v)\cdot\nabla\Phi(x,v)\right\} dv\,dx \\ &+ \int_{\mathbf{R}^{d-1}\times\mathbf{R}} \bar\rho^0(\bar x,v)\Phi(0,\bar x,v)\,dv\,d\bar x = \int_{\mathbf{\Pi}\times\mathbf{R}} \frac{\partial\Phi}{\partial v}(x,v)\,d\bar m(x,v)\end{aligned}$$

for all $\Phi \in \mathcal{D}(\mathbf{\Pi}\times\mathbf{R})$. Here $\bar m$ is a bounded non-negative measure defined on $\mathbf{\Pi}\times\mathbf{R}$ and $\bar\rho^0$ is the initial data. This equation is supplemented with an assumption on the structure of $\bar\rho$. If the function χ is defined by

$$(1.6)\qquad \chi(v|v_1,v_2) := \begin{cases} +1 & \text{if } v_2 < v < v_1 \\ -1 & \text{if } v_1 < v < v_2 \\ 0 & \text{otherwise} \end{cases}$$

for all $v_1, v_2 \in \mathbf{R}$, then $\bar\rho$ should have the form

$$(1.7)\qquad \bar\rho(x,v) = \chi(v|\bar u(x),0)$$

for some scalar function $\bar u$ defined on $\mathbf{\Pi}$ (An analogous statement should hold for the initial data). Then we have the following equivalence (shown in [LPT'94])

Theorem 1.1. *(i) Let $\bar u$ be a weak entropy solution of problem (1.1). Then there is a bounded non-negative measure $\bar m$ such that*

$$\bar\rho := \chi(\cdot|\bar u,0) \in L^1(\mathbf{\Pi}\times\mathbf{R}) \cap L^\infty(\mathbf{\Pi}, L^1(\mathbf{R}))$$

solves the transport equation (1.5) for appropriate initial data. The measure $\bar m$ is supported in $\mathbf{\Pi}\times[-\bar M,\bar M]$, where $\bar M := \|\bar u\|_{L^\infty(\mathbf{\Pi})}$. Furthermore, we have

$$\|\bar m\|_{\mathcal{M}(\mathbf{\Pi}\times\mathbf{R})} \le \frac{1}{2}\left\|\bar u^0\right\|^2_{L^2(\mathbf{R}^{d-1})}.$$

(ii) Vice versa, let $\bar\rho \in L^1(\mathbf{\Pi}\times\mathbf{R}) \cap L^\infty(\mathbf{\Pi}, L^1(\mathbf{R}))$ and a bounded non-negative measure $\bar m$ be given, that solve the transport problem (1.5). Assume that $\bar\rho$ can be written as in (1.7) for some function $\bar u$. Then $\bar u$ is a weak entropy solution of the Cauchy problem (1.1).

Note that by definition

$$\int_{\mathbf{R}} \chi(v|v_1, v_2)\varphi'(v)\, dv = \int_{v_2}^{v_1} \varphi'(v)\, dv = \varphi(v_1) - \varphi(v_2) \tag{1.8}$$

for sufficiently smooth φ.
There is a strong similarity between (1.5) and kinetic equations as e.g. the Boltzmann equation. From that perspective, $\bar{\rho}$ is some sort of distribution function, and assumption (1.7) means that $\bar{\rho}$ should be a "Maxwellian" (cf. Bouchut [Bo'98]).
The second important ingredient of our proof is a discrete entropy inequality (cf. theorem 2.5 below). Here, we estimate the rate of entropy dissipation over each cell in terms of the local oscillation of the numerical flux function. We refer to [KNR'95] and [No'95]. It turns out that this result fits very neatly into the kinetic formulation stated above.
Finally, our analysis relies on so-called velocity averaging lemmata first introduced by Golse, Lions, Perthame and Sentis [GLPS'88] (see also DiPerna, Lions and Meyer [DLM'91] and Bézard [Be'94]. For more recent results we refer to Bouchut [Bo'98] and the references therein). The velocity averaging technique allows to prove the strong compactness of a sequence of approximate solutions u^h of problem (1.2)/(1.4). The principal idea is that the macroscopic quantity $\bar{u}$ has *more* regularity than $\bar{\rho}$ whose v-average it is. The following result is a variant of theorem B in [LPT'94] adapted to the estimates we get in section 3

Theorem 1.2. *Let $1 < p \leq 2$ and $0 < \gamma < 1$. Choose some test function $\psi \in \mathcal{D}(\mathbf{R})$ and define $\Lambda := \operatorname{spt}\psi$. Assume there are sequences $(\rho^h), (m^h)$ and (π^h) uniformly bounded in $L^p(\mathbf{R}^d \times \Lambda), L^1(\Lambda, \mathcal{M}(\mathbf{R}^d))$ and $L^1(\Lambda, B^{1,1}_{-\gamma}(\mathbf{R}^d))$ respectively, such that*

$$\frac{\partial \rho^h}{\partial t} + f' \cdot \nabla \rho^h = \frac{\partial m^h}{\partial v} + \pi^h \quad \textit{in } \mathcal{D}'(\mathbf{R}^d \times \mathbf{R}). \tag{1.9}$$

If now the following non-degeneracy condition holds

$$\sup_{(\tau,\bar{\xi}) \in \mathbf{R}^d} \operatorname{meas}\left\{v \in \Lambda \colon \tau + f'(v) \cdot \bar{\xi} = 0\right\} = 0, \tag{1.10}$$

then the sequence $u^h = \int_{\mathbf{R}} \rho^h(\cdot, v)\psi(v)\, dv$ belongs to a compact subset of $L^1_{loc}(\mathbf{R}^d)$.

Remark 1.3. Here $L^1(\Lambda, X)$ stands for the space of strongly measurable, integrable functions on Λ taking values in X, where X is some Banach space (cf. [DU'77]). $\mathcal{M}(\mathbf{R}^d)$ is the space of bounded Radon measures, and $B^{1,1}_{-\gamma}(\mathbf{R}^d)$ is a Besov space (consult definition 3.3 below).

Note that the non-degeneracy condition (1.10) (which we will assume throughout) restricts the class of admissible flux functions: f should be nonlinear. Theorem 1.2 is another instance of the fact that the nonlinearity of a problem can have a regularizing effect on the solutions. Think of the transport operator $\partial_t + f' \cdot \nabla$ as a directional derivative along the vector $(1, f')$. Then the partial regularity information contained in equation (1.9) is transformed into compactness of the moments of ρ^h, that is of u^h, as long as a condition on the distribution of the directions $(1, f')$ holds true. This is the heart of the matter.
Condition (1.10) or some variant of it appears in many papers dealing with averaging lemmas (see e.g. [LPT'94] or [Bo'98]). It can be seen as a generalization of an assumption formulated by Tartar [Ta'83] in his existence proof for scalar conservation laws in one spatial dimension.

The structure of this paper is as follows: In the next section we define a class of finite volume schemes for the scalar conservation law (1.1) and state the main convergence result. This theorem is proved in section 3. In the last section we outline the proof of the velocity averaging result 1.2.

2. A class of finite volume schemes

Let I be a countable index set and (T_i), $i \in I$, a family of closed convex polygons $T_i \subset \mathbf{R}^{d-1}$. We assume that the T_i cover the whole space, and that the intersection of two different polygons consist of common faces and vertice only. Define the mesh parameter h as $\sup_i \operatorname{diam} T_i$. Let (S_{ij}) be the faces of T_i, (n_{ij}) the corresponding outer unit normal vectors and J_i their number. Then we have

$$\sum_{j=1}^{J_i} |S_{ij}|\, n_{ij} = 0.$$

By definition, for every T_i there is exactly one T_k with $T_i \cap T_k = S_{ij}$. We denote that polygon by T_{ij}. Next choose $0 =: t_0 < t_1 < \ldots < t_{N-1} < t_N := T$ and define $\Delta t^n := t^{n+1} - t^n$. Now the family of space-time prisms $T_i^n := \left[t^n, t^{n+1}\right] \times T_i$ for $n = 0 \ldots N-1$ and $i \in I$ gives an unstructured mesh on $\mathbf{\Pi}$. We write $S_i^n := \{t^n\} \times T_i$ for faces normal to the time direction, while faces in spatial directions are called S_{ij}^n. Finally, we denote the polygon neighboring T_i^n at the face S_{ij}^n by T_{ij}^n.
The finite volume approximation u^h of the entropy solution $\bar{u}$ will be piecewise constant on the cells of an unstructured mesh with mesh parameter h. To keep the notation simpler, we omit the index h in what follows. We write $u(x) =: u_i^n$ (resp. u_{ij}^n) for almost all $x := (t, \bar{x}) \in T_i^n$ (resp. T_{ij}^n). The update formula is given by

$$u_i^{n+1} = u_i^n - |T_i|^{-1} \sum_{j=1}^{J_i} \left|S_{ij}^n\right| g_{ij}^n \tag{2.1}$$

for some approximate flux function g_{ij}^n to be defined in a moment. The numbers

$$u_i^0 := \frac{1}{|T_i|} \int_{T_i} \bar{u}^0(\bar{x})\, d\bar{x} \tag{2.2}$$

are taken as numerical initial data. It is well known that in this case the sequence of approximate initial data converges strongly in $L^1_{loc}(\mathbf{R}^{d-1})$ to $\bar{u}^0$.
The class of approximate fluxes, to which the convergence result given below applies, is the class of so-called E-fluxes as introduced by Osher [Os'84]. An E-flux is a family $(g_{ij}), i \in I, j = 1 \ldots J_i$, of measurable functions $g_{ij} : \mathbf{R}^2 \longrightarrow \mathbf{R}$ which are conservative, consistent and satisfy Osher's condition E: For $v_1, v_2 \in \mathbf{R}$ and all $v \in [\min(v_1, v_2), \max(v_1, v_2)]$

$$\frac{g_{ij}(v_1, v_2) - f(v) \cdot n_{ij}}{v_1 - v_2} \geq 0.$$

One example of an E-flux is Godunov's flux

$$g_{ij}(v_1, v_2) := \begin{cases} \min\limits_{v_1 \leq v \leq v_2} f(v) \cdot n_{ij} & \text{if } v_1 \leq v_2 \\ \max\limits_{v_1 \geq v \geq v_2} f(v) \cdot n_{ij} & \text{if } v_1 > v_2, \end{cases}$$

another one is the Lax-Friedrichs flux. Every E-flux can be obtained from these two as a convex combination (cf. Tadmor [Td'84]). We will restrict ourselves to Godunov's flux in all what follows. Godunov's flux can be rewritten

$$g_{ij}(v_1, v_2) = f(w_{ij}(v_1, v_2)) \cdot n_{ij}. \tag{2.3}$$

Here (w_{ij}) is a family of piecewise continuous functions $w_{ij} : \mathbf{R}^2 \longrightarrow \mathbf{R}$ such that whenever $S_{ij} = S_{kl}, i \neq k$, then $w_{ij}(v_1, v_2) = w_{kl}(v_2, v_1)$ for all $v_1, v_2 \in \mathbf{R}$. Note that $w_{ij}(v, v) = v$ for $v \in \mathbf{R}$. Godunov's flux is Lipschitz-continuous and monotone, that means non-decreasing in the first and non-increasing in the second argument. In case of a first-order scheme, the approximate flux is now given by

$$g_{ij}^n := g_{ij}(u_i^n, u_{ij}^n). \tag{2.4}$$

It is also possible to consider higher-order schemes, but we will not do this here. The approximate entropy flux corresponding to the entropy U is defined as

$$G_{ij}(v_1, v_2) := F(w_{ij}(v_1, v_2)) \cdot n_{ij}$$

for all $v_1, v_2 \in \mathbf{R}$. Obviously, G_{ij} is consistent and conservative, too. Moreover, we have the compatibility relation

$$\partial_k G_{ij}(v_1, v_2) = U'(w_{ij}(v_1, v_2)) \partial_k g_{ij}(v_1, v_2). \tag{2.5}$$

Here ∂_k stands for the partial derivative with respect to the kth argument, $k = 1, 2$. We use the notation

$$G_{ij}^n := G_{ij}(u_i^n, u_{ij}^n).$$

Update formula (2.1) can be recast in a somewhat different form. We assume that we are given numbers $\Delta x_{ij}^n > 0$ such that

$$\sum_{j=1}^{J_i} |S_{ij}| \, \Delta x_{ij}^n = |T_i| \, .$$

Then we define $\lambda_{ij}^n := \Delta t^n / \Delta x_{ij}^n$ and $\sigma_{ij}^n := \Delta x_{ij}^n \, |S_{ij}| \, |T_i|^{-1}$. Now we can write

$$u_i^{n+1} = \sum_{j=1}^{J_i} \sigma_{ij}^n u_{ij}^{n+1}, \quad \text{with } u_{ij}^{n+1} := u_i^n - \lambda_{ij}^n (g_{ij}^n - f(u_i^n) \cdot n_{ij}). \tag{2.6}$$

We refer to Noelle [No'95] for a discussion of the numbers Δ_{ij}^n.

Theorem 2.1. *Let (u^h) be a sequence of approximate solutions of (1.1) built from the finite volume scheme described in the previous section. Assume that*
(i) The sequence (u^h) is uniformly bounded for all sufficiently small h

$$M := \sup_h \left\| u^h \right\|_{L^\infty(\mathbf{\Pi})} < \infty$$

and has uniformly compact support in Π.
(ii) There exists $s \in \left(\frac{1}{2}, 1\right)$ such that for $\Delta t := \inf_n \Delta t^n$

$$\lim_{h \to 0} h^{2s} / \Delta t = 0. \tag{2.7}$$

(iii) There exists a constant $\epsilon > 0$ such that for $\delta = \epsilon h / \sqrt{\Delta t}$ and all i, j, n

$$(1 + \delta) \lambda_{ij}^n L_{ij} \leq 1 \quad \text{where} \quad L_{ij} = \| f' \cdot n_{ij} \|_{L^\infty([-M,M])}.$$

(iv) The non-degeneracy condition (1.10) *holds.*
Then a subsequence of (u^h) converges strongly in $L^1_{loc}(\mathbf{\Pi})$ to a weak entropy solution of the Cauchy problem (1.1).

Remark 2.2. We first remark that convergence can be shown not only for Godunov's scheme, but for the whole class of E-schemes. It is also possible to treat *higher-order* schemes (see [NW'97]). Higher-order means that on each cell a polynomial reconstruction of the data is built using the numbers u_i^n at a given time level. Then the values of these reconstructions at fixed offpoints on the cell faces are used in definition (2.4) of the approximate fluxes g_{ij}^n. Since we need to assume that the higher-order correction terms can be estimated from above by some power of the mesh parameter h [NW'97], these schemes are still first-order only in the vicinity of discontinuities.

Remark 2.3. The assumptions we made in theorem 2.1 are comparable to those in Noelle [No'95], where DiPerna's convergence theory is used. However, the approach presented here relies on the nonlinearity of the flux, that is on assumption (1.10).

Remark 2.4. Uniform boundedness was shown in Cockburn, Hou and Shu [CHS'90] and Geiben [Ge'93]. We will not reprove this here. Since we assume compactly supported initial data, u^h will live on a bounded set for all schemes with a finite speed of propagation, e.g. for standard finite volume schemes with $\Delta t \geq Ch$ for some constant C not depending on h.

In the proof of theorem 2.1 the following discrete entropy inequality, which holds for Godunov's flux as well as for other E-schemes, plays a prominent role [No'95].

Theorem 2.5. *For all convex entropies $U \in C^2(\Lambda)$ and all i, j, n*

$$U(u_{ij}^{n+1}) - U(u_i^n) + \lambda_{ij}^n \left(G_{ij}^n - F(u_i^n) \cdot n_{ij}\right) + \delta \frac{\sigma}{2} (\lambda_{ij}^n)^2 \left(g_{ij}^n - f(u_i^n) \cdot n_{ij}\right)^2 \leq 0.$$

Here $\sigma := \min_{-M \leq v \leq M} U''(v)$.

3. Proof of Theorem 2.1 (Convergence)

The proof consists of two steps. First we construct an approximate distribution function ρ^h from the numerical solution u^h and apply the transport operator to it. We split the resulting term into three parts and give bounds for them in various norms. In the second step we use the velocity averaging result 1.2 to show strong compactness of the approximate solution u^h and complete the proof.

3.1. **Some estimates.** Let us start with the definition of the distribution function. To simplify the notation we omit the index h most of the time. Extending u by zero we have

$$\rho(x, v) := \chi(v|u(x), 0) = \sum_{n=0}^{N-1} \sum_{i \in I} \chi(v|u_i^n, 0)\, \mathbf{1}_{T_i^n}(x)$$

for almost all $x = (t, \bar{x}) \in \mathbf{R}^d$ and $v \in \mathbf{R}$. From the Gauss-Green theorem we get

$$\frac{\partial \rho}{\partial t} + f' \cdot \nabla \rho =: R, \tag{3.1}$$

where

$$R = \sum_{i \in I} \chi(\cdot|u_i^0, 0)\, dS_i^0 + \sum_{i \in I} \chi(\cdot|0, u_i^N)\, dS_i^N$$
$$+ \sum_{n=0}^{N-1} \sum_{i \in I} \Big\{ \chi(\cdot|u_i^{n+1}, u_i^n)\, dS_i^{n+1} + \frac{1}{2} \sum_{j=1}^{J_i} f' \cdot n_{ij} \chi(\cdot|u_{ij}^n, u_i^n)\, dS_{ij}^n \Big\}. \tag{3.2}$$

Here dS_i^n is the $(d-1)-$dimensional Hausdorff measure restricted to S_i^n i.e.

$$dS_i^n := \mathbf{1}_{S_i^n} d\mathcal{H}^{d-1}$$

(same for S_{ij}^n). Note that in our notation the contribution from some cell face S_{ij}^n is counted twice: S_{ij}^n is the jth face of the cell T_i^n, but also the lth face of some neighbouring cell T_k^n. We compensated that by the factor one half in (3.2). Now we split R into three parts

$$\begin{aligned} R_0 &= \sum_{i\in I} \chi(\cdot|u_i^0, 0)\, dS_i^0 + \sum_{i\in I} \chi(\cdot|0, u_i^N)\, dS_i^N \\ R_1 &= \sum_{n=0}^{N-1} \sum_{i\in I} \Big\{ \chi(\cdot|u_i^{n+1}, u_i^n) + \sum_{j=1}^{J_i} \sigma_{ij}^n \lambda_{ij}^n \eta_{ij}^n \Big\} dS_i^{n+1} \\ R_2 &= \sum_{n=0}^{N-1} \sum_{i\in I} \sum_{j=1}^{J_i} \eta_{ij}^n \left\{ dS_{ij}^n - \sigma_{ij}^n \lambda_{ij}^n\, dS_i^{n+1} \right\}, \end{aligned} \tag{3.3}$$

where

$$\eta_{ij}^n = f' \cdot n_{ij} \chi(\cdot|w_{ij}(u_i^n, u_{ij}^n), u_i^n).$$

To prove the identity $R = R_0 + R_1 + R_2$ we only have to check that

$$\frac{1}{2} \sum_{i\in I} \sum_{j=1}^{J_i} f' \cdot n_{ij} \chi(\cdot|u_{ij}^n, u_i^n)\, dS_{ij}^n = \sum_{i\in I} \sum_{j=1}^{J_i} f' \cdot n_{ij} \chi(\cdot|w_{ij}(u_i^n, u_{ij}^n), u_i^n) dS_{ij}^n.$$

But this follows easily from the properties of χ and w_{ij}: For fixed ij, let kl be the unique index pair defined by $S_{ij}^n = S_{kl}^n$ and $i \neq k$. Then $u_i^n = u_{kl}^n, u_{ij}^n = u_k^n$ and $n_{ij} = -n_{kl}$. Since $w_{kl}(v_1, v_2) = w_{ij}(v_2, v_1)$ and $\chi(\cdot|v_1, v_2) = -\chi(\cdot|v_2, v_1)$ for $v_1, v_2 \in \mathbf{R}$ (cf. section 2 and definition (1.6)) we have

$$\begin{aligned} n_{kl} \chi(\cdot|w_{kl}(u_k^n, u_{kl}^n), u_k^n) &= -n_{ij} \chi(\cdot|w_{kl}(u_{ij}^n, u_i^n), u_{ij}^n) \\ &= -n_{ij} \chi(\cdot|w_{ij}(u_i^n, u_{ij}^n), u_{ij}^n) \\ &= n_{ij} \chi(\cdot|u_{ij}^n, w_{ij}(u_i^n, u_{ij}^n)) \end{aligned}$$

Using $\chi(\cdot|v_1, v_2) + \chi(\cdot|v_2, v_3) = \chi(\cdot|v_1, v_3)$ a.e. for all $v_1, v_2, v_3 \in \mathbf{R}$ we arrive at

$$n_{kl} \chi(\cdot|w_{kl}(u_k^n, u_{kl}^n), u_k^n) + n_{ij} \chi(\cdot|w_{ij}(u_i^n, u_{ij}^n), u_i^n) = n_{ij} \chi(\cdot|u_{ij}^n, u_i^n).$$

This proves our claim. Let us take a closer look at the three parts of R. We have R_0 because we extended u from $\mathbf{\Pi}$ to $\mathbf{R}^d$. Note that the first summand in (3.3) contains the numerical initial data. The second term R_1 is a measure for the entropy production in the scalar conservation law. It corresponds to the RHS equation of (1.5). Finally R_2 is the residual. It measures the numerical error. In the following, we will write $\Lambda := [-M, M]$.

Lemma 3.1. *The R_0^h are uniformly bounded in $L^1(\Lambda, \mathcal{M}(\mathbf{R}^d))$.*

Proof. Measurability follows from the tensor product structure of R_0^h (cf. [DU'77]), and the boundedness is immediate from our assumptions on u^h. □

Lemma 3.2. *R_1^h can be written as*

$$R_1^h = \frac{\partial m^h}{\partial v} \quad \text{in } \mathcal{D}' \tag{3.4}$$

for some non-negative uniformly bounded measure m^h.

Proof. We suppress the mesh index h. Clearly, to obtain (3.4) we may simply integrate R_1 in the kinetic variable. Using overbars to indicate primitives, as in

$$\bar{\chi}(v|v_1, v_2) := \int_{-M}^{v} \chi(s|v_1, v_2)\, ds$$

for $v, v_1, v_2 \in \Lambda$, we arrive at the representation

$$m = \sum_{n=0}^{N-1} \sum_{i \in I} m_i^{n+1}\, dS_i^{n+1}, \quad \text{where } m_i^{n+1} = \bar{\chi}(\cdot|u_i^{n+1}, u_i^n) + \sum_{j=1}^{J_i} \sigma_{ij}^n \lambda_{ij}^n \bar{\eta}_{ij}^n.$$

Note that R_1 vanishes outside the interval $[-M, M]$. Therefore, $m_i^{n+1}(v) = 0$ for $v \leq -M$. But if $v \geq M$ we have (using (1.8) and (2.3)/(2.4))

$$\begin{aligned} m_i^{n+1}(v) &= \int_{-M}^{v} \left\{ \chi(s|u_i^{n+1}, u_i^n) + \sum_{j=1}^{J_i} \sigma_{ij}^n \lambda_{ij}^n f'(s) \cdot n_{ij} \chi(s|w_{ij}(u_i^n, u_{ij}^n), u_i^n) \right\} ds \\ &= u_i^{n+1} - u_i^n + \sum_{j=1}^{J_i} \sigma_{ij}^n \lambda_{ij}^n \left(g_{ij}^n - f(u_i^n) \cdot n_{ij} \right) \end{aligned}$$

which vanishes again because of (2.6). Note that $\sum_{i=1}^{J_i} \sigma_{ij}^n = 1$. We conclude that m is compactly supported in $\mathbf{R}^d \times [-M, M]$. Now let us fix i, n for a moment. We choose a test function $U \in C^2(\mathbf{R})$ which is convex on $[-M, M]$ (a convex entropy) and apply its second derivative to m_i^n. Integrating by part and using compatibility relation (1.3) (and (1.8) again) we find

$$\text{(3.5)} \quad -\int_{\mathbf{R}} m_i^{n+1}(v) U''(v)\, dv = U(u_i^{n+1}) - U(u_i^n) + \sum_{j=1}^{J_i} \sigma_{ij}^n \lambda_{ij}^n \left(G_{ij}^n - F(u_i^n) \cdot n_{ij} \right)$$

(Remember that m_i^{n+1} has compact support.) This quantity can be controlled using the discrete entropy inequality in theorem 2.5. In fact, from representation (2.6) and Jensen's inequality we obtain

$$(3.5) \leq \sum_{j=1}^{J_i} \sigma_{ij}^n \left\{ U(u_{ij}^{n+1}) - U(u_i^n) + \lambda_{ij}^n \left(G_{ij}^n - F(u_i^n) \cdot n_{ij} \right) \right\} \leq 0$$

So, if we choose a sequence of convex entropies U_k with

$$\|U_k''\|_{C(\mathbf{R})} \leq 1 \quad \text{and} \quad U_k'' \to \mathbf{1}_{\mathrm{spt} m_i^{n+1,-}} \text{ a.e.},$$

where $m_i^{n+1,-} := \min(0, m_i^{n+1})$, we find from the dominated convergence theorem

$$\begin{aligned} \left\| m_i^{n+1,-} \right\|_{L^1(\mathbf{R})} &= -\int_{\mathbf{R}} m_i^{n+1}(v) \mathbf{1}_{\mathrm{spt} m_i^{n+1,-}}(v)\, dv \\ &= -\lim_{k \to \infty} \int_{\mathbf{R}} m_i^{n+1}(v) U_k''(v)\, dv \leq 0 \end{aligned}$$

Since this holds for all i, n we conclude that m is a non-negative measure as claimed. To show the boundedness of m, note that if the entropy is given by $U(v) := \frac{1}{2}v^2$

$$
\begin{aligned}
\|m\|_{L^1(\Lambda,\mathcal{M}(\mathbf{R}^d))} &= \sum_{n=0}^{N-1}\sum_{i\in I}|S_i^{n+1}|\left\{\int_\Lambda m_i^{n+1}(v)\,dv\right\} \\
&= -\sum_{n=0}^{N-1}\sum_{i\in I}|T_i|\Big\{U(u_i^{n+1})-U(u_i^n)+\sum_{j=1}^{J_i}\sigma_{ij}^n\lambda_{ij}^n\left(G_{ij}^n-F(u_i^n)\cdot n_{ij}\right)\Big\}.
\end{aligned}
\tag{3.6}
$$

But for all index pairs such that $S_{ij} = S_{kl}, i \neq k$ we have

$$\sigma_{ij}^n\lambda_{ij}^n\,|T_i|\,G_{ij}^n = \Delta t^n\,|S_{ij}|\,G_{ij}^n = -\Delta t^n\,|S_{kl}|\,G_{kl}^n = -\sigma_{kl}^n\lambda_{kl}^n\,|T_k|\,G_{kl}^n$$

because the approximate entropy flux is conservative and only the outer unit normal vectors n_{ij} change sign when going from the index pair ij to kl. Hence

$$\sum_{i\in I}\sum_{j=1}^{J_i}\sigma_{ij}^n\lambda_{ij}^n\,|T_i|\,G_{ij}^n = 0.$$

Furthermore, we have

$$\sum_{j=1}^{J_i}\sigma_{ij}^n\lambda_{ij}^n n_{ij} = \Delta t^n\,|T_i|^{-1}\sum_{j=1}^{J_i}|S_{ij}|\,n_{ij} = 0. \tag{3.7}$$

Therefore the j-sum in (3.6) drops out if we sum over all cells. The remaining $U(u_i^n)$-terms however appear twice with alternating signs and therefore cancel out, too, except for those with $n = 0$ and $n = N$. Since the entropy U is non-negative we finally arrive at

$$\|m\|_{L^1(\Lambda,\mathcal{M}(\mathbf{R}^d))} \leq \sum_{i\in I}|T_i|U(u_i^0) \leq \frac{1}{2}\left\|\bar{u}^0\right\|^2_{L^2(\mathbf{R}^{d-1})} \leq C < \infty. \tag{3.8}$$

(We used (2.2) and Jensen's inequality.) The lemma is proved. □

Definition 3.3. Let $\varphi_0 \in \mathcal{D}(\mathbf{R}^d)$ be a non-negative radially symmetric test function which equals 1 on the ball $B(0,1)$ and vanishes outside $B(0,2)$. Define

$$\varphi_1(\xi) := \varphi_0(2^{-1}\xi) - \varphi_0(\xi) \qquad \text{and} \qquad \varphi_j(\xi) := \varphi_1(2^{-j+1}\xi)$$

for $\xi \in \mathbf{R}^d, j \geq 2$. Introduce the dyadic operators $S_j \colon \Phi \mapsto \mathcal{F}^{-1}\{\varphi_j\hat{\Phi}\}$. Then the Besov space $B_s^{p,q}(\mathbf{R}^d)$ with $s \in \mathbf{R}$ and $1 \leq p, q \leq \infty$ contains all tempered distributions on $\mathbf{R}^d$ such that the norm

$$\|\Phi\|_{B_s^{p,q}(\mathbf{R}^d)} := \left(\sum_{j=0}^{\infty}2^{jsq}\|S_j\Phi\|^q_{L^p(\mathbf{R}^d)}\right)^{1/q}$$

(modified if $q = \infty$) stays finite (for more details consult Triebel [Tr'83]).

Lemma 3.4. *Let $\gamma = (1+s)/2$. Then*

$$\lim_{h\to 0}\left\|R_2^h\right\|_{L^1(\Lambda,B^{1,1}_{-\gamma}(\mathbf{R}^d))} = 0.$$

Remark 3.5. Note that the Besov space $B^{1,1}_{-\gamma}(\mathbf{R}^d)$ can be identified with the topological dual of the closure of $\mathcal{D}(\mathbf{R}^n)$ in $C^\gamma(\mathbf{R}^d)$ (the space of Hölder continuous functions). A proof of this result can be found in [Tr'78] .

Proof. Again we suppress the index h. First we show that for all i, j, n

$$\left\| dS_{ij}^n - \sigma_{ij}^n \lambda_{ij}^n dS_i^{n+1} \right\|_{B_{-\gamma}^{1,1}(\mathbf{R}^d)} \leq C \sigma_{ij}^n \lambda_{ij}^n \left| S_i^{n+1} \right| h^\gamma. \tag{3.9}$$

We apply a test function $\phi \in C^\gamma(\mathbf{R}^d)$ and obtain by definition of $\sigma_{ij}^n, \lambda_{ij}^n$

$$\int_{\mathbf{R}^d} \phi(x) \left\{ dS_{ij}^n(x) - \sigma_{ij}^n \lambda_{ij}^n dS_i^{n+1}(x) \right\} \leq \sigma_{ij}^n \lambda_{ij}^n \left| S_i^{n+1} \right| \left(\phi_{ij}^n - \phi_i^{n+1} \right)$$

with ϕ_i^{n+1} and ϕ_{ij}^n the averages of ϕ over the cell faces S_i^{n+1} and S_{ij}^n. Then

$$\begin{aligned} \left| \phi_{ij}^n - \phi_i^{n+1} \right| &\leq \left| \widetilde{\phi}_{ij}^n - \widetilde{\phi}_i^{n+1} \right| + \left| S_{ij}^n \right|^{-1} \int_{S_{ij}^n} \left| \phi - \widetilde{\phi}_{ij}^n \right| d\mathcal{H}^{d-1} \\ &+ \left| S_i^{n+1} \right|^{-1} \int_{S_i^{n+1}} \left| \phi - \widetilde{\phi}_i^{n+1} \right| d\mathcal{H}^{d-1} \leq C h^\gamma \left\| \phi \right\|_{C^\gamma(\mathbf{R}^d)}, \end{aligned}$$

where $\widetilde{\phi}_i^{n+1}$ resp. $\widetilde{\phi}_{ij}^n$ is the evaluation of ϕ in the center of mass of S_i^n resp. of S_{ij}^n. Next, we must control the L^1-norm of η_{ij}^n. For an arbitrary $U \in C^1(\mathbf{R})$ we have

$$\int_{\mathbf{R}} \eta_{ij}^n(v) U'(v)\, dv = G_{ij}^n - F(u_i^n) \cdot n_{ij} = \int_{u_i^n}^{u_{ij}^n} \partial_2 G_{ij}(u_i^n, v)\, dv. \tag{3.10}$$

The first identity follows as above from the compatibility relation (1.3) and (1.8) (consult also section 2). For the second we used the consistency and Lipschitz-continuity of the approximate entropy flux G_{ij}. To proceed we now replace the derivative of G_{ij} by (2.5). Since Godunov's flux is non-increasing in the second argument, the derivative of g_{ij} has a sign and we can estimate

$$\begin{aligned} |(3.10)| &\leq \int_{u_i^n}^{u_{ij}^n} \left| U'(w_{ij}(u_i^n, v))\, \partial_2 g_{ij}(u_i^n, v)\, dv \right| \\ &\leq \|U'\|_{C(\mathbf{R})} \left| \int_{u_i^n}^{u_{ij}^n} \partial_2 g_{ij}(u_i^n, v)\, dv \right| = \|U'\|_{C(\mathbf{R})} \left| g_{ij}^n - f(u_i^n) \cdot n_{ij} \right| \end{aligned}$$

using the consistency of g_{ij} and (2.4). Note that we do not assume convexity for U. Since the measurability of η_{ij}^n is obvious we learn that for all indices i, j, n

$$\|\eta_{ij}^n\|_{L^1(\mathbf{R})} \leq \left| g_{ij}^n - f(u_i^n) \cdot n_{ij} \right|.$$

Now the norm of R_2 can be bounded by

$$\|R_2\|_{L^1(\Lambda, B_{-\gamma}^{1,1}(\mathbf{R}^d))} \leq C h^\gamma \sum_{n=0}^{N-1} \sum_{i \in I} \sum_{j=1}^{J_i} \sigma_{ij}^n \lambda_{ij}^n |S_i^{n+1}| |g_{ij}^n - f(u_i^n) \cdot n_{ij}|$$

and further, using the Cauchy-Schwarz inequality, by

$$C h^\gamma \sqrt{\sum_{n=0}^{N-1} \sum_{i \in I} |T_i| \sum_{j=1}^{J_i} \sigma_{ij}^n \chi_i^n} \sqrt{\sum_{n=0}^{N-1} \sum_{i \in I} |T_i| \sum_{j=1}^{J_i} \sigma_{ij}^n (\lambda_{ij}^n)^2 \left(g_{ij}^n - f(u_i^n) \cdot n_{ij} \right)^2},$$

where χ_i^n is the characteristic function of the set of indices i, n for which u_i^n is non-vanishing. Note that by assumption, the support of the numerical solution is uniformly bounded. These terms can be handled easily: First we have

$$C h^\gamma \sqrt{\sum_{n=0}^{N-1} \sum_{i \in I} |T_i| \sum_{j=1}^{J_i} \sigma_{ij}^n \chi_i^n} \leq C \left(\frac{h^{2\gamma}}{\Delta t} \right)^{1/2}.$$

Moreover, from the discrete entropy inequality (2.5) with $U(v) := \frac{1}{2}v^2$ we find

$$\sum_{j=1}^{J_i} \sigma_{ij}^n (\lambda_{ij}^n)^2 \left(g_{ij}^n - f(u_i^n)\cdot n_{ij}\right)^2$$
$$\leq \quad -\frac{2}{\delta}\sum_{j=1}^{J_i} \sigma_{ij}^n \Big\{ U(u_{ij}^{n+1}) - U(u_i^n) + \lambda_{ij}^n \left(G_{ij}^n - F(u_i^n)\cdot n_{ij}\right)\Big\}.$$

By definition, $\sum_{j=1}^{J_i} \sigma_{ij}^n = 1$ for all i, n. Therefore (cf. (2.6))

$$-\sum_{j=1}^{J_i} \sigma_{ij}^n U(u_{ij}^{n+1}) \leq U\left(\sum_{j=1}^{J_i} u_{ij}^{n+1}\right) = U(u_i^{n+1})$$

which is Jensen's inequality. We proceed as in the proof of lemma 3.2 (cf. (3.6)) and arrive at

$$\left\|R_2^h\right\|_{L^1(\Lambda, B^{1,1}_{-\gamma}(\mathbf{R}^d))} \leq C\left(\frac{h^{2\gamma}}{\Delta t}\frac{1}{\delta}\right)^{1/2} \|\bar{u}^0\|_{L^2(\mathbf{R}^{d-1})}.$$

Note that $1/\delta$ explodes as $h \to 0$. But

$$\left(\frac{h^{2\gamma}}{\Delta t}\frac{1}{\delta}\right)^{1/2} = \left(\frac{h^{4\gamma-2}}{\epsilon^2 \Delta t}\right)^{1/4} = \left(\frac{h^{2s}}{\epsilon^2\Delta t}\right)^{1/4}$$

for $\gamma = (1+s)/2$, so finally we obtain

$$\left\|R_2^h\right\|_{L^1(\Lambda, B^{1,1}_{-\gamma}(\mathbf{R}^d))} \leq C\left(\frac{h^{2s}}{\Delta t}\right)^{1/4}.$$

Using assumption (2.7) we are finished. □

Remark 3.6. We stop here for a moment to summarize what we have shown so far. Since (ρ^h) is uniformly bounded in L^∞ a subsequence converges weak* to some function $\bar{\rho}$. Associated to (ρ^h) there is a sequence (R^h) as defined above. Given $\phi \in \mathcal{D}(\mathbf{\Pi})$ and $U \in \mathcal{D}(\mathbf{R})$ we have

$$-\int_{\mathbf{\Pi}\times\mathbf{R}} \rho^h(x,v)U'(v)\left\{\frac{\partial\phi}{\partial t}(x) + f'(v)\cdot\nabla\phi(x)\right\} dv\,dx = \left\langle R^h, \phi\otimes U'\right\rangle,$$

where

$$\left\langle R^h, \phi\otimes U'\right\rangle = \left\langle R_0^h, \phi\otimes U'\right\rangle - \left\langle m^h, \phi\otimes U''\right\rangle + \left\langle R_2^h, \phi\otimes U'\right\rangle. \tag{3.11}$$

The first term on the RHS goes to

$$\int_{\mathbf{R}^{d-1}\times\mathbf{R}} \bar{\rho}^0(\bar{x}, v)U'(v)\phi(0,\bar{x})\,dv\,d\bar{x}, \quad \text{where } \bar{\rho}^0 := \chi(\cdot|\bar{u}^0, 0).$$

For the second, we have shown in lemma 3.2 that (m^h) is uniformly bounded and non-negative in the sense of measures. Extracting another subsequence if necessary we have $m^h \to \bar{m}$ weak* and $\bar{m} \geq 0$. The third term finally goes to zero in distributional sense (even in a somewhat stronger topology) as shown in lemma 3.4. Therefore the pair $(\bar{\rho}, \bar{m})$ solves the transport equation (1.5). (We refer also to [LPT'94] for a similar calculation.) What remains to be done is to prove that the nonlinear constraint (1.7) holds true for $\bar{\rho}$. For this we use the velocity averaging technique and show that (some subsequence of) (u^h) converges strongly in L^1_{loc}.

3.2. **End of Proof.** To apply theorem 1.2, we choose a test function ψ which equals 1 on the interval $[-M, M]$, and define $\pi^h := R_0^h + R_2^h$. Beacause of lemmas 3.1 and 3.4, and since the space of measure is continuously embedded into $B^{1,1}_{-\gamma}(\mathbf{R}^d)$ (cf. [Tr'83]), π^h satisfies the assumptions of theorem 1.2. Moreover

$$\|\rho^h\|_{L^p(\mathbf{R}^d\times\Lambda)} = \|u^h\|^{1/p}_{L^1(\mathbf{R}^d)}$$

which is uniformly bounded, too. But then theorem 1.2 shows that u^h belongs to a compact subset of $L^1_{loc}(\mathbf{R}^d)$. Since

$$\int_{\mathbf{R}} |\rho^{h_1}(\cdot,v) - \rho^{h_2}(\cdot,v)|\,dv = \int_{\mathbf{R}} |\chi(v|u^{h_1},u^{h_2})|\,dv = |u^{h_1} - u^{h_2}|$$

the approximate distribution function ρ^h converges strongly in $L^1_{loc}(\mathbf{R}^d\times\mathbf{R})$ (up to a subsequence). Hence, the nonlinear constraint (1.7) holds for the limit $\bar\rho$. From theorem 1.1 we conclude that $\bar u$ is a weak entropy solution.

Remark 3.7. One classical approach to proving strong compactness for sequences of approximate solutions consists in establishing a uniform bound on the total variation and then making use of Helly's theorem. For the more modern approach relying on measure-valued solutions as introduced by DiPerna no such control is necessary. Once one has shown consistency with the entropy condition, the L^1-contraction ensures compactness. The result presented in this paper lies somewhere in between these two cases. In fact, we do need some control over the residual, but this bound is comparatively easy to obtain, since we can choose a very weak topology.

4. Proof of Theorem 1.2 (Velocity Averaging)

For completeness, we would like to give an outline of proof for theorem 1.2. We will skip most details since the arguments are technically involved and can be found in other papers on velocity averaging.
Let us fix some test function $\psi \in \mathcal{D}(\mathbf{R})$ and denote the RHS of equation (1.9) by R^h. Then we can recover ρ^h from R^h (formally) by inverting the transport operator

$$\rho^h(\cdot,v) = \mathcal{F}^{-1}\left\{\frac{\hat R^h(\xi,v)}{i(\tau+f'(v)\cdot\bar\xi)}\right\} \quad \text{for all } v\in\mathbf{R}, \xi=(\tau,\bar\xi)\in\mathbf{R}^d \tag{4.1}$$

(the Fouriertransform is taken with respect to space-time only). But now we face the problem that the symbol $-i(\tau+f'(v)\cdot\bar\xi)^{-1}$ becomes unbounded. We will need a splitting. Let $\varphi\in\mathcal{D}(\mathbf{R})$ be a non-negative even test function, vanishing outside the interval $[-2,2]$, with $\varphi = 1$ on $[-1,1]$. Then we define two operators

$$A_\lambda\colon \Phi \longmapsto \mathcal{F}^{-1}\left\{\int_{\mathbf{R}} \varphi\left(\lambda^{-1}\frac{\tau+f'(v)\cdot\bar\xi}{|\xi|}\right)\hat\Phi(\xi,v)\psi(v)\,dv\right\} \tag{4.2}$$

for some parameter $\lambda\in(0,\infty)$, and

$$B_\lambda\colon \Phi \longmapsto \mathcal{F}^{-1}\left\{\int_{\mathbf{R}} \big(1-\varphi\big)\left(\lambda^{-1}\frac{\tau+f'(v)\cdot\bar\xi}{|\xi|}\right)\frac{\hat\Phi(\xi,v)}{i(\tau+f'(v)\cdot\bar\xi)}\psi(v)\,dv\right\}. \tag{4.3}$$

Note that the inverse symbol $-i(\tau+f'(v)\cdot\bar\xi)^{-1}$ appears in (4.3), but because of the cut-off function φ it is effective only in the region

$$\Big\{\xi=(\tau,\bar\xi)\in\mathbf{R}^d : \big|\tau+f'(v)\cdot\bar\xi\big| \geq \lambda|\xi|\Big\}, \tag{4.4}$$

that means outside a neighborhood around the singular set. Therefore it is reasonable to expect that B_λ has nice properties. Let $\Lambda := \operatorname{spt}\psi$. Then we have

Lemma 4.1. *There exists a constant C not depending on $\lambda \in (0,\infty)$ such that*

$$\|A_\lambda(\Phi)\|_{L^p(\mathbf{R}^d)} \le C\big(\eta(2\lambda)\big)^{1/p'}\|\Phi\|_{L^p(\mathbf{R}^d\times\Lambda)} \tag{4.5}$$

for all $\Phi \in L^p(\mathbf{R}^d\times\Lambda), 1 < p \le 2$. The function η is given by

$$\eta(\lambda) := \sup_{(\tau,\bar\xi)\in\mathbf{R}^d} \operatorname{meas}\big\{v\in\Lambda\colon \big|\tau + f'(v)\cdot\bar\xi\big| \le \lambda|\xi|\big\}.$$

Remark 4.2. We assumed that the non-degeneracy condition (1.10) holds. It is easy to show that in that case $\eta(\lambda)\to 0$ as $\lambda\to 0$. As a consequence, the function $A_\lambda(\Phi)$ for suitable Φ becomes small in L^p-norm if we let λ go to zero.

Definition 4.3. The generalized (fractional) Sobolev space $H^p_s(\mathbf{R}^d)$ is defined for $1\le q\le\infty, s\in\mathbf{R}$ as the space of all tempered distributions such that the norm

$$\|\Phi\|_{H^p_s(\mathbf{R}^d)} := \|(\mathrm{Id}-\Delta)^{s/2}\Phi\|_{L^p(\mathbf{R}^d)}$$

stays finite. For more details consult [Tr'83].

Lemma 4.4. *Let $1<p\le 2, \lambda\in(0,\infty)$. Then we have for all $\Phi\in L^p(\mathbf{R}^d\times\Lambda)$*

$$\|B_\lambda(\Phi)\|_{H^p_1(\mathbf{R}^d)} \le C_\lambda\|\Phi\|_{L^p(\mathbf{R}^d\times\Lambda)}. \tag{4.6}$$

C_λ grows as $\lambda\to 0$. The same estimate holds for the operator $B'_\lambda : \Phi\mapsto B_\lambda\left(\frac{\partial\Phi}{\partial v}\right)$.

These two lemmas are shown as in [DLM'91] (compare (4.5)/(4.6) with the estimates (22)/(23) in that paper) with the modifications explained in the appendix of [LPT'94]. Note that the operators B_λ, B'_λ are smoothing: We gain one derivative. We will now prove theorem 1.2 from these two results. First we note that the dyadic operators S_j playing a role in the definition 3.3 of the Besov spaces commute both with B_λ and B'_λ. We may therefore rewrite inequality (4.6) using Besov norms (and Minkowski's inequality) with $(\mathrm{Id}-\Delta)^{1/2}B_\lambda(\Phi)$ instead of $B_\lambda(\Phi)$ on the LHS. But the operator $(\mathrm{Id}-\Delta_x)^{1/2}$ defines an isomorphism (a lifting) between Besov spaces of different regularity (cf. Triebel [Tr'83]). We conclude that

$$\|B_\lambda(\Phi)\|_{B^{p,q}_{1+s}(\mathbf{R}^d)} \le C_\lambda\|\Phi\|_{L^1(\Lambda,B^{p,q}_s(\mathbf{R}^d))} \tag{4.7}$$

for all $1<p\le 2, 1\le q\le\infty$ and $s\in\mathbf{R}$. The same holds for B'_λ. Now for some $\lambda\in(0,\infty)$ we have a splitting

$$u^h = A_\lambda(\rho^h) + \big\{-B'_\lambda(m^h) + B_\lambda(\pi^h)\big\}. \tag{4.8}$$

Denote by $u_0^{\lambda,h}$ the first term on the RHS of (4.8), and by $u_1^{\lambda,h}$ the terms in brackets. As already pointed out in remark 4.2

(4.9) $u_0^{\lambda,h}$ can be made arbitrarily small in $L^1_{loc}(\mathbf{R}^d)$ uniformly with respect to h by choosing λ small enough.

Moreover we have

(4.10) $u_1^{\lambda,h}$ is strongly compact in $L^1_{loc}(\mathbf{R}^d)$ for all λ.

To see this, we choose a p near 1 such that the number $\gamma' := \gamma + d/p'$ is less than 1 (which is always possible since $\gamma<1$) and use the continuous embedding

$$\mathcal{M}(\mathbf{R}^d) \hookrightarrow B^{1,1}_{-\gamma}(\mathbf{R}^d) \hookrightarrow B^{1,\infty}_{-\gamma'}(\mathbf{R}^d) \tag{4.11}$$

(cf. [Tr'83]) to show that

$$m^h, \pi^h \text{ are uniformly bounded in } L^1(\Lambda, B^{1,\infty}_{-\gamma'}(\mathbf{R}^d)).$$

We conclude from (4.7) that $u_1^{\lambda,h}$ is uniformly bounded in some Besov space with strictly positive regularity, and therefore relatively compact in $L^1_{loc}(\mathbf{R})$. But then the same is true for the sequence (u^h). This proves our claim.

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