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Finite energy solutions to the isentropic Euler equations

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We consider the isentropic Euler equations of gas dynamics in one space dimension. They take the form of a system of hyperbolic conservation laws

$$\left.\begin{aligned} \partial_t \varrho + \partial_x(\varrho u) &= 0 \\ \partial_t(\varrho u) + \partial_x(\varrho u^2 + P) &= 0 \end{aligned}\right. \quad \text{in } [0,\infty) \times \Omega, \tag{1.1}$$

with suitable initial data $(\varrho, u)(t = 0, \cdot) =: (\bar{\varrho}, \bar{u})$ to be specified later. For the moment, let us consider the case $\Omega = \mathbf{R}$. The first equation in (1.1), called the continuity equation, decribes the local conservation of mass. Here $\varrho(t, x) \geqslant 0$ is the mass density at time $t \in [0, \infty)$ and location $x \in \mathbf{R}$, and $u(t, x) \in \mathbf{R}$ is the Eulerian velocity. The second equation in (1.1), called the momentum equation, models the local conservation of momentum.

The physics of the system is reflected in the equation of state, which determines the pressure P as a function of the thermodynamical quantities. Under the simplifying assumption of entropy being constant in space and time, the pressure is a function of the density only. We will consider the special case of polytropic gases in the following, for which

$$P(r) := \kappa r^\gamma \quad \text{for all } r \geqslant 0, \tag{1.2}$$

where $\gamma > 1$ is the adiabatic coefficient and $\kappa > 0$ is some constant. Physically relevant choices are $\gamma = 1 + 2/n$, where $n \in \mathbf{N}$ is the number of degrees of freedom of the gas molecules. The choice $\gamma = 2$ plays a role in shallow water models of compressible flows, where the pressure term in (1.1) represents the influence of gravitation on the fluid layer.

A useful normalization, which we will assume throughout the paper, is

$$\kappa := \frac{\theta^2}{\gamma}, \quad \theta := \frac{\gamma - 1}{2}, \quad \lambda := \frac{3 - \gamma}{2(\gamma - 1)}. \tag{1.3}$$

The significance of the constant λ will be explained later. An interesting special case (though of limited physical interest) is $\gamma = 3$, for which $\lambda = 0$.

Let us integrate the continuity equation against the characteristic function $\mathbf{1}_{(a,b)}$ of some interval. Using the Gauss-Green formula we find that

$$\begin{aligned}0 &= \int_{\mathbf{R}} \Big(\partial_t \varrho + \partial_x(\varrho u)\Big)(t,x) \mathbf{1}_{(a,b)}(x)\,dx \\ &= \frac{d}{dt}\int_a^b \varrho(t,x)\,dx + \Big(\partial_x(\varrho u)(t,b) - (\varrho u)(t,a)\Big)\end{aligned}$$

(formally). The integral on the right-hand side represents the amount of mass contained in the interval (a,b) at time t. The second term is the flux. The fact that the first equation in (1.1) is in divergence form in space and time therefore implies the well-known conservation property: the change of mass contained in the interval (a,b) as time evolves equals the flow of mass through the boundary of the interval. Letting $a \to \infty$ and $b \to -\infty$ (and assuming sufficiently fast decay of the flux at infinity), we obtain in particular that

$$\frac{d}{dt}\int_{\mathbf{R}} \varrho(t,x)\,dx = 0. \tag{1.4}$$

Since $\varrho(t,x) \geqslant 0$ the integral in (1.4) is equal to the norm $\|\varrho(t,\cdot)\|_{\mathscr{L}^1(\mathbf{R})}$, which gives us a first a priori bound that all reasonable solutions of (1.1) must satisfy: If the mass is finite initially, then it remains finite at all positive times. In fact, the total mass is constant in time. A similar remark applies to the total momentum (the integral of $(\varrho u)(t,x)$ over $\mathbf{R}$). But since the momentum does not have a sign, the conservation of total momentum is not as useful.

Note that formally we can use the chain rule in (1.1) to obtain

$$\begin{aligned}0 &= \partial_t(\varrho u) + \partial_x\big(\varrho u^2 + P(\varrho)\big) \\ &= \Big(\partial_t \varrho + \partial_x(\varrho u)\Big)u + \varrho\Big(\partial_t u + u\partial_x u\Big) + \partial_x P(\varrho).\end{aligned}$$

The first term on the right-hand side vanishes because of the continuity equation. Dividing by ϱ we find that wherever $\varrho > 0$ it holds

$$\partial_t u + u\partial_x u + \frac{1}{\varrho}\partial_x P(\varrho) = 0 \quad \text{in } [0,\infty)\times\Omega \tag{1.5}$$

(neglecting boundary terms). We have that $\frac{1}{\varrho}\partial_x P(\varrho) = \partial_x U'(\varrho)$, with internal energy $U(\varrho)$ related to the pressure by the formula

$$P(r) = U'(r)r - U(r) \quad \text{for all } r \geqslant 0.$$

The quantity $U'(\varrho)$ is called the enthalpy. For polytropic gases we have

$$U(r) = \frac{\kappa}{\gamma - 1} r^\gamma \quad \text{for all } r \geqslant 0.$$

Formally, the system of isentropic Euler equations (1.1) is therefore equivalent to the continuity equation augmented by the velocity equation

$$0 = \partial_t u + u\,(\partial_x u) + \partial_x U'(\varrho) \quad \text{in } [0,\infty) \times \Omega. \tag{1.6}$$

Unlike the system (1.1), equation (1.6) is no longer in divergence form.

Again arguing formally, we can rewrite the continuity equation in the form

$$\partial_t \varrho + u\,(\partial_x \varrho) + \varrho\,(\partial_x u) = 0,$$

which upon multiplication by $U'(\varrho)$ yields

$$\begin{aligned} 0 &= \partial_t U(\varrho) + u\,\partial_x U(\varrho) + U'(\varrho)\varrho\,(\partial_x u) \\ &= \partial_t U(\varrho) + \partial_x\big(U(\varrho)u\big) + \underbrace{\big(U'(\varrho)\varrho - U(\varrho)\big)}_{=P(\varrho)}(\partial_x u). \end{aligned} \tag{1.7}$$

On the other hand, when multiplying (1.5) by ϱu we obtain

$$0 = \partial_t\big(\tfrac{1}{2}\varrho u^2\big) + \partial_x\Big(\big(\tfrac{1}{2}\varrho u^2 + P(\varrho)\big)u\Big) - P(\varrho)\,(\partial_x u). \tag{1.8}$$

Summing up (1.7) and (1.8) we find a new conservation law for the energy:

$$0 = \partial_t\big(\tfrac{1}{2}\varrho u^2 + U(\varrho)\big) + \partial_x\Big(\big(\tfrac{1}{2}\varrho u^2 + Q(\varrho)\big)u\Big), \tag{1.9}$$

where $Q(r) := U(r) + P(r)$ for all $r \geqslant 0$. The term $\frac{1}{2}\varrho u^2$ is called the kinetic energy and $U(\varrho)$ is the internal energy. Arguing as we did for the continuity equation, we obtain another a priori bound that any realistic solution of (1.1) must satisfy: If the total energy is bounded initially, then it remains bounded for all positive times. In fact, formally we obtain from (1.9) that

$$\frac{d}{dt}\int_{\mathbf{R}} \big(\tfrac{1}{2}\varrho u^2 + U(\varrho)\big)(t,x)\,dx = 0. \tag{1.10}$$

Notice, however, that this identity was derived under the assumption that the solution consists of functions that are smooth in space and time. But it is well known that solutions of hyperbolic conservation laws do not remain smooth, even if the initial data is arbitrary regular. On the contrary, solutions typically form jump discontinuities along codimension-one manifolds, which are called shocks. In the context of gas dynamics, a shock may represent for example the sudden increase in pressure that we hear as a sonic boom of an airplane passing by. Mathematically, the correct way to handle the nonsmoothness of solutions (which makes sense from a physical point of view as well) is to relax (1.9) to a differential inequality: replace the $=$ by an $\leqslant$. As a consequence, we obtain an inequality in (1.10) too. Physically, this means that energy may transition into a form that is not taken into account in the model, such as heat. From the point of view of the model, such energy is being lost.

Another consequence of the fact that solutions of hyperbolic conservation laws are nonsmooth, is that the solution concept must be adjusted: one must consider weak solutions (the partial differential equations are to be satisfied in the sense of distributions). To understand what kind of information (1.1) entails, let us consider a typical situation. There exists a (sufficiently) regular curve in space-time, along which the conserved quantities $(\varrho, \varrho u)$ of (1.1) are discontinuous; see Figure 1. We denote by s the local speed of the curve, which

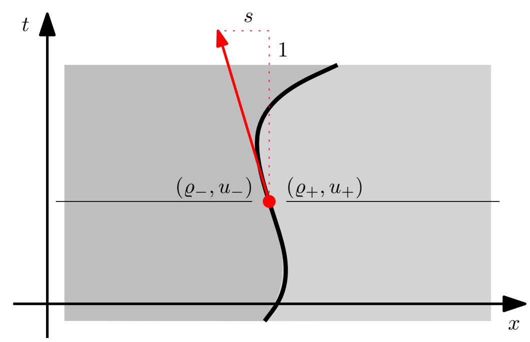

Fig. 1.1. Solutions may be discontinuous along shocks.

is called the shock speed. We further assume that for each time t the left and right limits of $(\varrho, \varrho u)(t, x)$ exist as x approaches a point on the curve. We indicate the left and right limits by a subscript $-$ resp. $+$. If $(\varrho, \varrho u)$ satisfy the conservation laws (1.1) in the distributional sense, then the Rankine-Hugoniot condition must be satisfied for each point (t, x) along the curve:

$$-s[\varrho] + [\varrho u] = 0, \tag{1.11}$$

$$-s[\varrho u] + [\varrho u^2 + P(\varrho)] = 0. \tag{1.12}$$

Here $[\varrho] := \varrho_+ - \varrho_-$ etc. We rewrite identity (1.11) in the form

$$-s\varrho_+ + \varrho_+ u_+ = -s\varrho_- + \varrho_- u_- =: J.$$

Since $u_+ - s$ and $u_- - s$ is the relative speed of the fluid compared to the speed of the shock, the number J describes the amount of mass that is flowing into the shock from either side. Obviously, conservation of mass requires that whatever mass flows into the shock on one side must flow out of the shock on the opposite side. We can simplify the identity (1.12), writing

$$P_+ - P_- = -u_+\big(\varrho_+(u_+ - s)\big) + u_-\big(\varrho_-(u_- - s)\big) = -(u_+ - u_-)J.$$

Whenever $J \neq 0$, then any change in the pressure (equivalently, in the density) across the shock must be accompanied by a change in velocity, and vice versa. Notice that if the characteristics are parallel to the shock, thus $u_+ = u_- = s$, then $J = 0$, which implies that $P_+ = P_-$ and $\varrho_+ = \varrho_-$. Consequently, there is no discontinuity at all. (This is different with more than one space dimension.) But if $u_+ \neq u_-$, then also $\varrho_+ \neq \varrho_-$. One can check that it is not possible to have a discontinuity with the density on one side being equal to zero.

In this paper, we are concerned with the question of existence of solutions to the isentropic Euler equations (1.1) globally in time. This question has been studied extensively before in the literature. The first global existence result was found by DiPerna [9] for the special values of $\gamma = 1 + 2/n$ with n an even integer, using the compensated compactness method, which we will explain in more detail below. His result was generalized in [1, 3, 7, 8, 15, 16]. Global existence of spherically symmetric solutions, which are basically one dimensional, was considered by Glimm and Chen [2]. To avoid the difficulty of such solutions becoming potentially unbounded near the origin, they constructed solutions outside a ball around the origin only. The case of spherically symmetric flows in the whole space, including the origin where multi-dimensional effects may occur, was treated in LeFloch and Westdickenberg [13].

The general strategy for constructing global solutions to the isentropic Euler equations (1.1) consists of three steps: First, find a sequence of approximate solutions, which are usually solutions of an approximation to the problem:

- *Discretization.* By substituting difference quotients for the differential operators one obtains finite difference schemes. The approximate solutions generated this way depend on a small parameter like the grid length. This approach allows us to compute approximate solutions numerically. More sophisticated discretization leads to the Lax-Friedrichs scheme or to finite-volume and finite-element schemes.
- *Parabolic Regularization.* By adding a higher-order differential operator on the right-hand side of (1.1) one obtains parabolic equations, for which existence of solutions can often be established. A simple example is
$$\left.\begin{aligned} \partial_t \varrho_\varepsilon + \partial_x(\varrho_\varepsilon u_\varepsilon) &= \varepsilon \partial_{xx} \varrho_\varepsilon \\ \partial_t(\varrho_\varepsilon u_\varepsilon) + \partial_x\big(\varrho_\varepsilon u_\varepsilon^2 + P(\varrho_\varepsilon)\big) &= \varepsilon \partial_{xx}(\varrho_\varepsilon u_\varepsilon) \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbf{R},$$
which is basically a perturbation of the heat equation. Here again $\varepsilon > 0$ is a small parameter. A more physical choice is to approximate the isentropic Euler equations by the Navier-Stokes equation:
$$\left.\begin{aligned} \partial_t \varrho_\varepsilon + \partial_x(\varrho_\varepsilon u_\varepsilon) &= 0 \\ \partial_t(\varrho_\varepsilon u_\varepsilon) + \partial_x\big(\varrho_\varepsilon u_\varepsilon^2 + P(\varrho_\varepsilon)\big) &= \varepsilon \varrho_\varepsilon^\alpha \, \partial_{xx} u_\varepsilon \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbf{R}, \tag{1.13}$$
for some parameter $\alpha \geqslant 0$. Since now the parabolic regularization only acts on the momentum, existence of solutions to (1.13) is highly nontrivial. We refer the reader to [10, 14] for more details.

Other possibilities include relaxation, kinetic approaches, and more.

The second step to prove existence of solutions of (1.1) consists in showing that the sequence $\{(\varrho_\varepsilon, \varrho_\varepsilon u_\varepsilon)\}_{\varepsilon>0}$ is relatively compact in a suitable space, for example some Lebesgue space. Then it is possible to extract a subsequence that converges with respect to some topology to a limit $(\varrho, \varrho u)$.

Finally, pass to the limit and show that $(\varrho, \varrho u)$ is a solution of (1.1).

A number of issues arise when trying to implement the three-step strategy outlined above. The first one involves the possibility of vacuum: Note that the conserved quantities in (1.1) are the density ϱ and the momentum $m := \varrho u$. Rewriting the equations in terms of (ϱ, m) we obtain the system

$$\left.\begin{aligned} \partial_t \varrho + \partial_x m &= 0 \\ \partial_t m + \left(\frac{m^2}{\varrho} + P(\varrho)\right) &= 0 \end{aligned}\right\} \quad \text{in } [0,\infty) \times \mathbf{R}. \tag{1.14}$$

A problem may thus arise when $\varrho = 0$, that is in the vacuum. In order for all terms in the momentum equation in (1.14) to be well-defined it is necessary that m vanishes whenever ϱ vanishes. Thinking of ϱ and m as measures, we require that m is absolutely continuous with respect the ϱ. Then u is uniquely determined as the Radon-Nikodym derivative of m with respect to ϱ.

On a more technical level, the possible occurence of vacuum causes problems since the strict hyperbolicity of the hyperbolic system (1.1) disappears when ϱ approaches zero. The two characteristic speeds coincide.

The second issue is related to the question of regularity. To establish the relative compactness of a sequence of functions, one typically needs uniform bounds and some form of uniform regularity. Consider for example the classical compactness criterion for sequences of continuous functions $f_n \in \mathscr{C}(I)$ with I some closed interval, the Arzelà-Ascoli theorem: If the sequence $\{f_n\}_n$ is

1. uniformly bounded
2. equicontinuous

(which is true if the f_n are differentiable with uniformly bounded derivatives, for example), then $\{f_n\}_n$ is relatively compact with respect to convergence in the sup-norm. Assumptions like equicontinuity are not easy to check.

Asking for convergence in $\mathscr{L}^1_{\text{loc}}$ only (instead of uniform convergence) does not improve the situation since still some form of uniform regularity must be established to obtain relative compactness. Since solutions of (1.1) typically have jump discontinuities along codimension-one submanifolds, a natural approach is to look for uniform bounds in the space BV of functions with bounded variation (functions whose distributional derivatives are measures), but uniform BV-bounds are typically not available either.

What can be done?

There are two points to keep in mind: First, the relative compactness can be with respect to an arbitrarily weak topology *provided* we can show that

$$\big(\varrho_\varepsilon u_\varepsilon^2, P(\varrho_\varepsilon)\big) \rightharpoonup \big(\varrho u^2, P(\varrho)\big)$$

(along a suitable subsequence). In other words, the only thing that matters is that we can pass to the limit in the nonlinear quantities that appear in the conservation laws (1.1). Then we can show that $(\varrho, \varrho u)$ is a solution.

Second, it is possible to derive additional regularity information by looking at nonlinear quantities of $(\varrho_\varepsilon, \varrho_\varepsilon u_\varepsilon)$ and the partial differential equations they satisfy. As discussed above, it is natural to require that solutions of the isentropic Euler equations satisfy an additional differential inequality

$$\partial_t\big(\tfrac{1}{2}\varrho u^2 + U(\varrho)\big) + \partial_x\Big(\big(\tfrac{1}{2}\varrho u^2 + Q(\varrho)\big)u\Big) \leqslant 0 \tag{1.15}$$

in the distributional sense, which expresses the physically reasonable assumption that no energy can be created. Since a nonpositive distribution is a measure and since the left-hand side of (1.15) contains differential operators, we obtain an additional piece of information about the regularity of $(\varrho, \varrho u)$.

We remark in passing that it seems to be kind of counterintuitive to impose additional requirements on solutions (so-called entropy conditions) like the one that (1.15) be satisfied in distributional sense: we do not even know whether solutions exist at all! But it turns out that the right set of entropy conditions actually helps in proving existence of solutions.

We proceed as we did before and try to find functions $\eta(\varrho, u)$ and $q(\varrho, u)$ with the property that if $(\varrho, \varrho u)$ is a smooth solution of (1.1), then

$$\partial_t\eta(\varrho, u) + \partial_x q(\varrho, u) = 0.$$

Since a priori we do not know anything about the particularities of the solution, we must cover all possible values of ϱ and u. The chain gives

$$\eta'(\varrho, u)\,\partial_t\varrho + (\partial_u\eta)(\varrho, u)\,\partial_t u + q'(\varrho, u)\,\partial_x\varrho + (\partial_u q)(\varrho, u)\,\partial_x u = 0.$$

We use the $'$ to indicate the derivative with respect to ϱ. Note that

$$\begin{aligned} \partial_t\varrho &= -u(\partial_x\varrho) - \varrho(\partial_x u), \\ \partial_t u &= -u(\partial_x u) - U''(\varrho)(\partial_x\varrho), \end{aligned}$$

because of the continuity and velocity equations. Therefore

$$\begin{aligned} &\Big[-\eta'(\varrho, u)u + q'(\varrho, u) - (\partial_u\eta)(\varrho, u)U''(\varrho)\Big](\partial_x\rho) \\ &\quad + \Big[-(\partial_u\eta)(\varrho, u)u + (\partial_u q)(\varrho, u) - \eta'(\varrho, u)\varrho\Big](\partial_x u) = 0. \end{aligned}$$

This equation is certainly true for all values of ϱ and u provided that

$$\begin{aligned} q'(r, v) &= \eta'(r, v)v + (\partial_v\eta)(r, v)U''(r) \\ \partial_v q(r, v) &= \eta'(r, v)r + (\partial_v\eta)(r, v)v \end{aligned} \qquad \text{for all } r \geqslant 0 \text{ and } v \in \mathbf{R}. \tag{1.16}$$

The set of differential equations (1.16) are called compatibility conditions, and any pair (η, q) satisfying them is called an entropy-entropy flux pair. It is now possible to eliminate from (1.16) the entropy flux q by differentiating the first equation by v, the second one by r, and setting both terms equal:

$$\begin{aligned} &(\partial_v\eta')(r, v)v + \eta'(r, v) + (\partial_{vv}\eta)(r, v)U''(r) \\ &\quad = \eta''(r, v)r + \eta'(r, v) + (\partial_v\eta')(r, v)v, \end{aligned}$$

which after simplification yields the linear wave equation

$$\eta''(r,v) = \frac{U''(r)}{r}\partial_{vv}\eta(r,v) \quad \text{for all } r \geqslant 0 \text{ and } v \in \mathbf{R}, \tag{1.17}$$

where r plays the role of the time and v is the spatial variable. Notice that the factor $U''(r)/r \geqslant 0$ for all $r \geqslant 0$, for our choice of internal energy. To obtain solutions of (1.17), we must specify initial data at "time" $r = 0$. We consider the family of weak entropies, which are characterized by

$$\eta(r=0,v) = 0, \quad \eta'(r=0,v) = \psi(v) \quad \text{for all } v \in \mathbf{R}, \tag{1.18}$$

for suitable functions ψ. One can check that solutions to (1.17) and (1.18) exist and can be written as the convolution of g against the entropy kernel

$$\chi(s|r,v) := \Big(r^{2\theta} - (s-v)^2\Big)_+^\lambda \quad \text{for all } s \in \mathbf{R} \tag{1.19}$$

(recall the definition (1.3)). The function χ is in fact the fundamental solution of (1.17) with $\chi(s|r=0,v) = 0$ and $\chi'(s|r=0,v) = \delta_s(v)$, the Dirac measure at position $s \in \mathbf{R}$. We stress the fact that χ is the null function if $r = 0$.

For any function ψ on $\mathbf{R}$ we now obtain a corresponding entropy

$$\eta_\psi(r,v) := \int_{\mathbf{R}} \psi(s)\chi(s|r,v)\,ds \quad \text{for all } r \geqslant 0 \text{ and } v \in \mathbf{R}. \tag{1.20}$$

One can show that η_ψ is convex as a function of r and *the momentum* rv if ψ is convex. We have the following identities for all $r \geqslant 0$ and $s, v \in \mathbf{R}$:

$\psi(s)$	1	s	$\frac{1}{2}s^2$
$\eta_\psi(r,v)$	r	rv	$\frac{1}{2}rv^2 + U(r)$

(1.21)

A similar representation is available for the entropy fluxes. We have

$$q_\psi(r,v) = \int_{\mathbf{R}^d} \psi(s)\sigma(s|r,v)\,ds \quad \text{for all } r \geqslant 0 \text{ and } v \in \mathbf{R}, \tag{1.22}$$

for any function ψ on $\mathbf{R}$ and with entropy flux kernel σ given by

$$\sigma(s|r,v) := \Big[(1-\theta)v + \theta s\Big]\chi(s|r,v) \quad \text{for all } s \in \mathbf{R}. \tag{1.23}$$

Note that the entropy kernel χ and the corresponding entropy flux kernel differ only by a multiplication by a function. The factor is a linear combination of the macroscopic velocity v and the kinetic velocity s.

Following the strategy outlined above, one can now introduce an entropy condition as follows: among all weak solutions of (1.1) we are interested only in those that satisfy in addition the family of differential inequalities

$$\partial_t \eta_\psi(\varrho,u) + \partial_x q_\psi(\varrho,u) \leqslant 0 \tag{1.24}$$

in the distributional sense, for all functions ψ on $\mathbf{R}$ that are convex and quadratic at infinity. These inequalities are called entropy inequalities and we will refer to solutions of (1.1) satisfying (1.24) for all admissible ψ as entropy solutions. Note that we restricted ourselves to function ψ such that

$$|\psi(s)| \leqslant C(1+s^2) \quad \text{for all } s \in \mathbf{R},$$

the reason being that this way we can control the entropy η_ψ by the total energy; see (1.21). This is a natural choice. Notice further that any distribution with a sign is a measure, so (1.24) contains additional information about the regularity of the solution (ϱ,u). Formally (1.24) implies that

$$\frac{d}{dt}\int_{\mathbf{R}} \eta_\psi(t,x)\,dx \leqslant 0$$

in the distributional sense, so the total entropy at any positive time is bounded above by the initial entropy. Let us immediately emphasize one subtlety of our result: In this paper, we will discuss the existence of entropy solutions. Our solutions will have finite total energy at each time (corresponding to the entropy η_ψ with $\psi(s) = \frac{1}{2}s^2$ for all $s \in \mathbf{R}$). But our solutions will only satisfy the entropy inequalities (1.24) for all convex and *subquadratic* ψ:

$$\lim_{s\to\pm\infty} \frac{\psi(s)}{1+s^2} = 0.$$

The a priori bounds we can derive are not sufficient to pass to the limit in the entropy flux when ψ grows quadratically at infinity; see Remark 1.7.

Let us pause for a moment to gather what we know so far.

1. **Bounds.** We expect that solutions of (1.1) satisfy

$$\left.\begin{aligned} \int_{\mathbf{R}} \varrho(t,x)\,dx &\leqslant M \\ \int_{\mathbf{R}} \big(\tfrac{1}{2}\varrho u^2 + U(\varrho)\big)(t,x)\,dx &\leqslant E \end{aligned}\right\} \quad \text{for a.e. } t \geqslant 0, \tag{1.25}$$

   for suitable constants $M, E > 0$ representing the initial total mass and energy resp. The first inequality in (1.25) implies a uniform $\mathscr{L}^1(\mathbf{R})$-bound for $\varrho(t,\cdot)$. Since we consider polytropic gases, we even have $\varrho(t,\cdot) \in \mathscr{L}^\gamma(\mathbf{R})$ uniformly in time. For the velocity we find that (uniformly)

$$u(t,\cdot) \in \mathscr{L}^2(\mathbf{R}, \varrho(t,\cdot)) \quad \text{for a.e. } t \geqslant 0.$$

2. **Regularity.** Since we are interested in entropy solutions of (1.1), we control the divergences of certain nonlinear functions of (ϱ,u):

$$\partial_t \eta_\psi(\varrho, u) + \partial_x q_\psi(\varrho, u) \text{ is a finite measure}$$

for a large class of convex weight functions ψ on $\mathbf{R}$. It is not obvious what kind of regularity this information implies. It is most likely not enough to establish BV-bounds, for example. For a discussion of the maximal regularity one can obtain for scalar conservation laws, see [6].

We emphasize once more that the regularity information we obtain from the entropy inequalities is fairly weak. We are going to show that this is nevertheless enough to prove strong precompactness of suitable approximate solutions, and thus global existence of entropy solutions of (1.1).

Remark 1.1. Formally the isentropic Euler equations can be decoupled into two transport equations: Let (ϱ, u) be a solution of (1.1) and define

$$\overline{z} := u + \varrho^\theta \quad \text{and} \quad \underline{z} := u - \varrho^\theta. \tag{1.26}$$

The functions $\overline{z}$ and $\underline{z}$ are called Riemann invariants. Now note that

$$\begin{aligned}
\partial_t \overline{z} &= \partial_t u + \theta \varrho^{\theta-1} \partial_t \varrho \\
&= \Big(-u\partial_x u - \tfrac{1}{2}\theta \partial_x \varrho^{2\theta} \Big) + \theta \varrho^{\theta-1} \Big(-u\partial_x \varrho - \varrho \partial_x u \Big) \\
&= \Big(-u\partial_x u - \theta \varrho^\theta \partial_x \varrho^\theta \Big) + \Big(-u\theta \partial_x \varrho^\theta - \theta \varrho^\theta \partial_x u \Big) \\
&= -(u + \theta\varrho^\theta) \partial_x (u + \varrho^\theta);
\end{aligned}$$

see (1.3). A similar computation can be done for $\partial_t \underline{z}$. Defining

$$\overline{w}(\overline{z}, \underline{z}) := u + \theta \varrho^\theta \quad \text{and} \quad \underline{w}(\overline{z}, \underline{z}) := u - \theta \varrho^\theta$$

(recall identities (1.26)), we find that the Riemann invariants satisfy

$$\partial_t \overline{z} + \overline{w}(\overline{z}, \underline{z}) \partial_x \overline{z} = 0 \quad \text{and} \quad \partial_t \underline{z} + \underline{w}(\overline{z}, \underline{z}) \partial_x \underline{z} = 0.$$

Remark 1.2. In one space dimension one can construct solutions (ϱ, u) to the system of isentropic Euler equations (1.1) that are uniformly bounded in $\mathscr{L}^\infty$ (assuming of course that the initial data is uniformly bounded). These bounds are stronger than the ones we listed above: finite total mass and energy. Our bounds are much more natural, and they are the only (known) bounds that still work if we introduce some geometry into the problem. Consider

$$\left.\begin{aligned}
\partial_t(\varrho A) + \partial_x(\varrho u A) &= 0 \\
\partial_t(\varrho u A) + \partial_x(\varrho u^2 A) + A\partial_x P(\varrho) &= 0
\end{aligned}\right\} \text{ in } [0,\infty) \times \Omega, \tag{1.27}$$

for suitable $\Omega \subset \mathbf{R}$ and some function $A\colon \Omega \longrightarrow \mathbf{R}$. We look at two cases:

1. **Spherically symmetric flows.** We want to solve the multi-dimensional isentropic Euler equations, under the particular assumption that the solutions are invariant under rotations. In this configuration, the problem is essentially one-dimensional. Then we have $\Omega := (0, \infty)$ (the independent variable $x \in \Omega$ now represents the distance from the origin) and
$$A(x) := \omega_d x^{d-1} \quad \text{for all } x \in \Omega,$$
with $d > 1$ the space dimension and ω_d the area of the unit sphere in $\mathbf{R}^d$. One can imagine (we are not aware of a rigorous proof for this conjecture, though) that a spherically symmetric wave travelling towards the origin creates an unbounded density when hitting zero. In fact, this significant increase in density, and thus pressure, is what makes inertial confinement fusion work, and what lets stars collapse under their own gravitational field to form a black hole. A uniform $\mathscr{L}^\infty$-bounds for solutions of (1.27) does not appear very plausible in this geometric setting. On the other hand, the total mass and total energy – now defined as
$$\int_\Omega \varrho(t,x) A(x)\, dx \quad \text{and} \quad \int_\Omega \big(\tfrac{1}{2}\varrho u^2 + U(\varrho)\big)(t,x) A(x)\, dx$$
for a.e. $t \geqslant 0$ – are still finite if that is true for the initial data.
2. **Nozzle flows.** We consider flows through a pipe with varying cross section, whose diameter is given by the function $A\colon \Omega \longrightarrow [\underline{a}, \overline{a}]$ with $\Omega := \mathbf{R}$ and suitable constants $0 \leqslant \underline{a} < \overline{a}$. Again it is difficult to prove uniform $\mathscr{L}^\infty$-bounds for solutions of (1.27) because of the influence of the geometry. If the flow is forced through a region where the cross section of the pipe is small, then the density will increase. But again the natural bounds on total mass and total energy hold without any difficulties.

For the modified system (1.27) we can reuse all entropy-entropy flux pairs we have already identified for the flat case (1.1). We just have to slightly adapt the entropy inequalities, which now take the form
$$\partial_t\Big(\eta_\psi(\varrho,u)A\Big) + \partial_x\Big(q_\psi(\varrho,u)A\Big) + \Big(\varrho u \eta'_\psi(\varrho,u) - q_\psi(\varrho,u)\Big)\partial_x A \leqslant 0$$
in distributional sense. Here $'$ indicates the derivative with respect to ϱ. In the following, we will only consider the flat case with $A =$ constant for simplicity. Moreover, instead of actually studying the existence problem (which would require finding a suitable approximation) we will only consider the question of *stability*: We will assume that we already have a sequence of entropy solutions of (1.1) satisfying some uniform bounds to be specified, and then show that we can extract a subsequence that converges to another entropy solution of the isentropic Euler equation. This approach allows us to highlight how to establish strong convergence and how to pass to the limit in the nonlinear quantities, without being slowed down by the details of the particular approximation we are using to construct the sequence in the first place. The presentation is based on [13] to which we refer the reader for details.

1.1 Compensated Compactness

The technical tool that allows us to prove strong convergence of sequences of (approximate) solutions to the isentropic Euler equations (1.1) is the method of compensated compactness, which was introduced in the seventies by Murat and Tartar [17, 20] and has been spectacularly successful in a host of situations since. In this section, we will try to motivate and explain the compensated compactness method. We follow loosely the presentation in [5].

Let us start by looking at two examples, which highlight the difficulties that arise in weakly convergent sequences of functions. The first example is

$$f_n(x) := \sin(nx) \quad \text{for all } x \in [0,1] \text{ and } n \in \mathbf{N}.$$

For any fixed n the function is bounded pointwise by one and smooth, but as $n \to \infty$ the f_n oscillate faster and faster. There are uniform bounds, e.g.

$$f_n \in \mathscr{L}^1([0,1]) \quad \text{and} \quad \|f_n\|_{\mathscr{L}^1([0,1])} \leqslant 2$$

for all n. But we do not have any uniform regularity for the sequence. The derivative $f_n'(x) = n\cos(nx)$, for example, becomes large in any Lebesgue norm as $n \to \infty$. One can then show that $\{f_n\}_n$ does not converge strongly in any $\mathscr{L}^p([0,1])$-norm with $1 \leqslant p \leqslant \infty$. The sequence does, however, converge *weakly.* The idea behind weak convergence is that instead of computing a norm, we simply test the sequence against a suitable family of test functions. That is, we integrate the f_n against some function φ and check the convergence of the corresponding sequence of integrals, which is a sequence of numbers. One can show that *for all* $\varphi \in \mathscr{L}^\infty([0,1])$ we have

$$\int_0^1 f_n(x)\varphi(x)\,dx \longrightarrow 0. \tag{1.28}$$

We say that the sequence $\{f_n\}_n$ converges weakly in $\mathscr{L}^1([0,1])$. The weak limit is the zero function. There are different weak convergences. They differ by what the set of test functions is. Typically the uniform bounds one can establish for the sequence of functions limits what is the right space of test functions. After all, the integrals in (1.28) must be defined.

Let us consider some subset $\Omega \subset \mathbf{R}^d$ and some exponent $1 < p < \infty$. We say that a sequence of functions $f_n \in \mathscr{L}^p(\Omega)$ converges weakly to $f \in \mathscr{L}^p(\Omega)$ (we describe this fact using the short-hand notation $f_n \rightharpoonup f$) provided

$$\int_\Omega f_n(x)\varphi(x)\,dx \longrightarrow \int_\Omega f(x)\varphi(x)\,dx \quad \text{for all } g \in \mathscr{L}^q(\Omega),$$

where q is conjugate exponent defined by $1/p + 1/q = 1$. Note that the Hölder inequality guarantees that the integrals are indeed well-defined. The weak convergence in $\mathscr{L}^p(\Omega)$ implies that the sequence $\{f_n\}_n$ is uniformly bounded in $\mathscr{L}^p(\Omega)$, which therefore is a necessary condition. In some sense, boundedness is also sufficient: One can show that for any sequence $\{f_n\}_n$ of functions that are uniformly bounded in $\mathscr{L}^p(\Omega)$, there exists a subsequence (still denoted by $\{f_n\}_n$ for simplicity) and some function $f \in \mathscr{L}^p(\Omega)$ such that $f_n \rightharpoonup f$. Thus uniform boundedness implies weak precompactness. No regularity is required. This extremely useful observation follows from the Banach-Aalaoglu theorem and the fact that for $1 < p < \infty$, the Lebesgue spaces are reflexive.

The situation is a bit more complicated for $p = 1$ or $p = \infty$, since then the reflexity of $\mathscr{L}^p(\Omega)$ is lost. One can define a notion of weak convergence for sequences in $\mathscr{L}^\infty(\Omega)$ (which is called weak* convergence) by testing against all functions on $\mathscr{L}^1(\Omega)$. One still obtains precompactness of uniformly bounded sequences in $\mathscr{L}^\infty(\Omega)$. For $p = 1$, however, uniform boundedness is not sufficient to obtain weak convergence, as our second example demonstrates:

$$g_n(x) := n\mathbf{1}_{(0,1/n)} \quad \text{for all } x \in [0,1] \text{ and } n \in \mathbf{N}.$$

The sequence $\{g_n\}_n$ is uniformly bounded in $\mathscr{L}^1([0,1])$ (the norm of each g_n is equal to one), and the g_n converge to zero for all x. So the zero function is again the natural candidate for the weak limit. If we choose $\varphi(x) := 1$ for all $x \in [0,1]$ (which clearly is in $\mathscr{L}^\infty([0,1])$, then we find that

$$\int_0^1 g_n(x)\varphi(x)\,dx \longrightarrow 1 \neq 0$$

as $n \to \infty$. The difficulty in this example is that the sequence $\{g_n\}_n$ exhibits a concentration. The g_n converge to an object that is no longer a Lebesgue measurable function but a measure (namely the Dirac measure located at the origin). Therefore uniform boundedness in $\mathscr{L}^1(\Omega)$ is typically not sufficient to have precompactness and weak convergence (up to a subsequence) to a limit element in $\mathscr{L}^1(\Omega)$. An additional assumption is needed to rule out possible concentrations. It is sufficient, for example, to have uniform boundedness in $\mathscr{L}^p(\Omega)$ with $p > 1$ (the functions g_n above are not uniformly bounded in any $\mathscr{L}^p([0,1])$ if $p > 1$). Alternatively, one can consider weak convergence in the space $\mathscr{M}(\Omega)$ of all finite measures instead, which is defined by testing against continuous functions (that vanish at infinity in case Ω is unbounded). This convergence is called weak* in the sense of measures. Since $\mathscr{L}^1(\Omega) \subset \mathscr{M}(\Omega)$, any uniformly bounded sequence $\{g_n\}_n$ in $\mathscr{L}^1(\Omega)$ admits a subsequence that converges weak* towards a finite measure. This is often enough.

The problem with weak convergent sequences of functions is that they do not play nicely with compositions with other functions. Let us again consider the sequence of functions f_n defined above. We have that $f_n \rightharpoonup 0$ weakly in any $\mathscr{L}^p([0,1])$ for $1 \leqslant p < \infty$, or weak* in $\mathscr{L}^\infty([0,1])$, or weak* in the sense of measures. But what happens if we consider a new sequence of functions h_n, defined as ccompositions $h_n := \phi \circ f_n$ of f_n with a suitable function ϕ? Even if $\{h_n\}_n$ converges weakly to some limit, in general we have that

$$\text{weak limit of } \{h_n\}_n \neq \phi\Big(\text{weak limit of } \{f_n\}_n\Big).$$

For example, the sequence of functions $h_n(x) := (f_n(x))^2$ converges weakly to the constant function $1/2$, which is different from the weak limit of $\{f_n\}_n$ (the zero function) squared. Weak convergence therefore does not commute with compositions of functions, unless the function is linear.

There exists a tool that captures the behavior of a weakly convergent sequence and thus allows us to represent the weak limit of compositions of the sequence with nonlinear functions. This tool is called a Young measure. We first consider a special situation. Let $\Omega \subset \mathbf{R}^d$ be an open set and consider a uniformly bounded sequence of functions $U^k \in \mathscr{L}^\infty(\Omega, \mathbf{R}^m)$. Extracting a subsequence if necessary, we may assume that $U^k \rightharpoonup \bar{U}$ weak* in $\mathscr{L}^\infty(\Omega, \mathbf{R}^m)$ (testing against $\mathscr{L}^1$-functions) for suitable $\bar{U} \in \mathscr{L}^\infty(\Omega, \mathbf{R}^m)$. If ϕ is any continuous function on $\mathbf{R}^m$, then the sequence of compositions $\phi(U^k)$ (contains a subsequence that) converges to $\bar{\phi}$ weak* in $\mathscr{L}^\infty(\Omega)$. Typically, we have $\bar{\phi} \neq \phi(\bar{U})$. But there exists a (weakly measurable) map

$$\nu \colon \Omega \longrightarrow \mathscr{P}(\mathbf{R}^m),$$

with $\mathscr{P}(\mathbf{R}^m)$ the space of Borel probability measures on $\mathbf{R}^m$, with the property that for all continuous functions $\phi \colon \mathbf{R}^m \longrightarrow \mathbf{R}$ we have

$$\phi(U^k) \rightharpoonup \int_{\mathbf{R}^m} \phi(y)\, \nu_\cdot(dy) =: \langle \phi \rangle \quad \text{weak* in } \mathscr{L}^\infty(\Omega). \tag{1.29}$$

The assumption of weak measurability of ν ensures that the map $x \mapsto \langle \phi \rangle(x)$ for $x \in \Omega$ is Lebesgue measurable and an element in $\mathscr{L}^\infty(\Omega)$. The Young measure is therefore capable of capturing the local statistics of possible oscillations in the weakly convergent sequence $\{U^k\}_k$ in such a way that the limit of compositions can be represented. In fact, one can show that

$$\nu_x = \lim_{r\to 0} \lim_{k\to\infty} \fint_{B_r(x)} \delta_{U^k(\xi)}\, d\xi \quad \text{for a.e. } x \in \Omega.$$

Note that the integral in the right-hand side is a convex combination of probability measures, which is again in $\mathscr{P}(\mathbf{R}^m)$. We emphasize that once we have the Young measure it is no longer necessary to extract subsequences. For *any* continuous function ϕ we know the weak* limit the $\phi(U^k)$ converge to.

The proof of this result relies again on the Banach-Alaoglu theorem and weak* precompactness. We sketch the argument. Let $K \subset \mathbf{R}^m$ be a compact set that contains the range of U^k for all k. Such a compact set exists because the sequence $\{U^k\}_k$ is assumed to be uniformly bounded in $\mathscr{L}^\infty(\Omega, R^m)$. We then consider the maps $\nu^n \colon \Omega \longrightarrow \mathscr{P}(K)$ defined by

$$\nu^k_x := \delta_{U^k(x)} \quad \text{for a.e. } x \in \Omega, \tag{1.30}$$

with δ_y the Dirac measure located at $y \in \mathbf{R}^m$. Then the sequence

$$\{\nu^k\}_k \text{ is uniformly bounded in } \mathscr{L}^\infty_{\mathrm{w}}(\Omega, \mathscr{P}(K)),$$

the space of weakly measurable and essentially bounded maps from Ω into the space of probability measures on K. This space is the topological dual of the space of absolutely integrable maps from Ω into $\mathscr{C}(K)$:

$$\Big(\mathscr{L}^1(\Omega, \mathscr{C}(K))\Big)^* = \mathscr{L}^\infty_{\mathrm{w}}(\Omega, \mathscr{P}(K)),$$

and so by Banach-Alaoglu theorem one can extract a weak* converging subsequence (testing against elements in $\mathscr{L}^1(\Omega, \mathscr{C}(K))$) whose limit is the Young measure. Then (1.29) follows from the definition of weak* convergence.

Here is a quick example: Let $\Omega := [0,1]$ and consider $U^k(x) := \sin(kx)$ as above. Then the corresponding Young measure is given by

$$\nu_x(dy) = \frac{dy}{\pi\sqrt{1-y^2}} \quad \text{for all } x \in \Omega.$$

One can check that the integral with respect to y over $\mathbf{R}$ is indeed one, so ν_x is a probability measure. Choosing $\phi(y) = y$ for all $y \in \mathbf{R}$, we obtain that the sequence $\{U^k\}_k$ converges weak* in $\mathscr{L}^\infty(\Omega)$ towards

$$\langle \phi \rangle(x) = \int_{\mathbf{R}} y\, \nu_x(dy) = 0 \quad \text{for all } x \in \Omega,$$

by symmetry. The choice $\phi(y) = y^2$ for all $y \in \mathbf{R}$ yields

$$\langle \phi \rangle(x) = \int_{\mathbf{R}} y^2\, \nu_x(dy) = \frac{1}{2} \quad \text{for all } x \in \Omega,$$

thus $\{(U^k)^2\}_k$ converges weak* in $\mathscr{L}^\infty(\Omega)$ to the constant function $1/2$. One can show that $\phi(U^k) \rightharpoonup \phi(\bar{U})$ weak* for *all* $\phi \in \mathscr{C}(K)$ if and only if

$$\nu_x = \delta_{\bar{U}(x)} \quad \text{for a.e. } x \in \Omega. \tag{1.31}$$

In this case, we even have strong convergence in $\mathscr{L}^p_{\mathrm{loc}}(\Omega)$ for any $p < \infty$ since (1.31) implies the convergence $U^k \longrightarrow \bar{U}$ in measure, which in combination with the dominated convergence theorem gives convergence in the norm.

Remark 1.3. Since the sequence $\{U^k\}_k$ is uniformly bounded in $\mathscr{L}^\infty(\Omega, \mathbf{R}^m)$, we can find a *compact* subset $K \subset \mathbf{R}^m$ that contains the range of all U^k. The compactness of K ensures that the Young measure ν, which is obtained as a weak* limit of the measures in (1.30), is again a probability measure for a.e. $x \in \Omega$. Otherwise, it may happen that $\nu_x(K) < 1$ for some $x \in \Omega$.

It we want to generalize the Young measure concept to sequences that are only uniformly bounded in $\mathscr{L}^p(\Omega, \mathbf{R}^m)$ with $p < \infty$, then it can become necessary to *compactify* the range of U^k for all k. That is, to embed $\mathbf{R}^m$ into a larger set that is compact. We may, for example, construct a compactification of $\mathbf{R}^m$ (which is obviously not compact) by adding a single point representing infinity and by introducing a suitable topology on this larger set that is consistent on $\mathbf{R}^m$ (in terms of convergence of point sequences) with the usual topology induced by the Euclidean norm. This compactification is called the Alexandroff or one-point compactification. Other choices are possible.

We already mentioned that sequences that are only uniformly bounded in $\mathscr{L}^1(\Omega, \mathbf{R}^m)$ may converge weak* towards a limit that is no longer a Lebesgue measurable function, but a measure. It is possible to construct Young measures that are capable of capturing not only the oscillations, but the concentrations as well. We are not going to use these constructions. Instead we will only use the fact that for any uniformly bounded sequence $\{U^k\}_k$ in $\mathscr{L}^1(\Omega, \mathbf{R}^m)$ there exists (after extracting a subsequence if necessary) an element

$$\nu \in \mathscr{L}^\infty_{\mathrm{w}}(\Omega, \mathscr{P}(\mathbf{R}^m))$$

(still called a Young measure) with the property that

$$\phi(U^k) \rightharpoonup \langle \phi \rangle = \int_{\mathbf{R}^m} \phi(y)\, \nu.(dy)$$

weakly in $\mathscr{L}^1(\Omega, \mathbf{R}^m)$ for all $\phi \in \mathscr{C}(\mathbf{R}^m)$ for which the sequence

$$\{\phi(U^k)\}_k \text{ is equi-integrable.} \tag{1.32}$$

The additional condition (1.32) precisely rules out the occurrence of concentrations. As was already mentioned before, the uniform boundedness of $\phi(U^k)$ in some $\mathscr{L}^p(\Omega)$ with $p > 1$ is sufficient for (1.32) to hold.

Weak convergence of sequences and composition with nonlinear functions are incompatible in general. It may happen, however, that a special structure of the nonlinearity allows us to pass to the limit anyway. The prime example of this kind of phenomenon is the so-called *div-curl-lemma*. Consider the following situation: Let $\Omega \subset \mathbf{R}^d$ be open and let $\{G^k\}_k$ and $\{H^k\}_k$ be two uniformly bounded sequences of functions in $\mathscr{L}^2(\Omega, \mathbf{R}^d)$. Assume that

$$\begin{matrix} G^k \rightharpoonup \bar{G} \\ H^k \rightharpoonup \bar{H} \end{matrix} \quad \text{weakly in } \mathscr{L}^2(\Omega, \mathbf{R}^d). \tag{1.33}$$

We assume some limited regularity on G^k and H^k in the form

$$\begin{matrix} \{\operatorname{div} G^k\}_k \\ \{\operatorname{curl} H^k\}_k \end{matrix} \quad \text{are precompact in } \mathscr{H}^{-1}(\Omega, \mathbf{R}^d). \tag{1.34}$$

Here $\mathscr{H}^{-1}(\Omega, \mathbf{R}^d)$ denotes the usual negative Sobolev space, the dual of the space $\mathscr{H}^1_0(\Omega, \mathbf{R}^d)$ of functions in $\mathscr{L}^2(\Omega, \mathbf{R}^d)$ vanishing at the boundary of Ω, whose distributional derivatives are square integrable. Note that (1.34) is not sufficient to show that either sequence $\{G^k\}_k$ or $\{H^k\}_k$ converges strongly in $\mathscr{L}^2(\Omega, \mathbf{R}^d)$. We have, however, the following weak continuity

$$G^k \cdot H^k \rightharpoonup \bar{G} \cdot \bar{H} \tag{1.35}$$

in the distributional sense. The existence result we are about to present is based on the convergence (1.35). The basic idea behind its proof is to cleverly rewrite the scalar product. Note first that without loss of generality, we may assume that $\bar{H}$ vanishes. Otherwise we replace H^k by $H^k - \bar{H}$ and use

$$G^k \cdot \bar{H} \rightharpoonup \bar{G} \cdot \bar{H}$$

weakly in $\mathscr{L}^2(\Omega, \mathbf{R}^d)$, by assumption (1.33) and the definition of weak convergence. We then consider solutions Φ^k of the Laplace equation

$$\begin{aligned} \Delta\Phi^k &= H^k \quad && \text{in } \Omega, \\ \Phi^k &= 0 && \text{in } \partial\Omega. \end{aligned}$$

By elliptic regularity theory, we find that

$$\Phi^k \in \mathscr{H}_0^1(\Omega, \mathbf{R}^d) \cap \mathscr{W}_{\mathrm{loc}}^{2,2}(\Omega, \mathbf{R}^d),$$

and the corresponding norms are uniformly bounded. Then we have

$$\begin{aligned} \Phi^k &\rightharpoonup 0 \quad && \text{weakly in } \mathscr{W}_{\mathrm{loc}}^{2,2}(\Omega, \mathbf{R}^d), \\ \operatorname{div} \Phi^k &\rightharpoonup 0 && \text{weakly in } \mathscr{H}_{\mathrm{loc}}^{1}(\Omega, \mathbf{R}^d), \\ \operatorname{curl} \Phi^k &\longrightarrow 0 && \text{strongly in } \mathscr{H}_{\mathrm{loc}}^{1}(\Omega, \mathbf{R}^d). \end{aligned}$$

The third statement follows from the identity $\operatorname{curl} \Phi^k = \Delta^{-1}(\operatorname{curl} H^k)$ and the assumption (1.34). We now define $V^k := H^k - \operatorname{grad} \operatorname{div} \Phi^k$. Then

$$V^k = \operatorname{div} \operatorname{curl} \Phi^k \longrightarrow 0 \quad \text{strongly in } \mathscr{L}_{\mathrm{loc}}^{2}(\Omega, \mathbf{R}^d).$$

We can then rewrite the scalar product $G^k \cdot H^k$ in the form

$$\begin{aligned} G^k \cdot H^k &= G^k \cdot V^k + G^k \cdot \operatorname{grad} \operatorname{div} \Phi^k \\ &= G^k \cdot V^k + \operatorname{div} \Big(G^k (\operatorname{div} \Phi^k) \Big) - (\operatorname{div} G^k)(\operatorname{div} \Phi^k). \end{aligned}$$

Now notice that sequences of products of two functions, one of which converges weakly, the other one converges strongly, converge in the distributional sense to the expected limits. Since weak convergence in $\mathscr{H}_{\mathrm{loc}}^{1}(\Omega, \mathbf{R}^d)$ implies strong convergence in $\mathscr{L}_{\mathrm{loc}}^{2}(\Omega, \mathbf{R}^d)$ one can check that

$$\begin{aligned} G^k \cdot V^k &\rightharpoonup 0 \\ G^k (\operatorname{div} \Phi^k) &\rightharpoonup 0 \quad \text{in the distributional sense,} \\ (\operatorname{div} G^k)(\operatorname{div} \Phi^k) &\rightharpoonup 0 \end{aligned}$$

from which the weak continuity (1.35) follows.

1.2 Finite Energy Solutions

In [13] we proved global existence of solutions to the isentropic Euler equations (1.1) assuming only the natural bounds of finite total mass and energy. We developed a compactness framework that allowed us to consider flows where geometric effects play an important role, such as nozzle flows or the multi-dimensional case with spherical symmetry. In this section, we will outline the main steps of the proof. Some details will be provided in the following sections. Let us start with a rigorous definition of the solution concept.

Definition 1.4. *Let initial data $(\bar{\varrho}, \bar{u})$ be given with finite mass and total energy. A pair of measurable functions $(\varrho, u) : [0,\infty)\times\mathbf{R} \longrightarrow [0,\infty)\times\mathbf{R}$ is called an entropy solution with finite mass and energy (or a finite energy solution, for short) to the isentropic Euler equations* (1.1) *if the following is true:*

1. *The total mass is conserved in time:*
$$\int_{\mathbf{R}} \varrho(t,x)\,dx = \int_{\mathbf{R}} \bar{\varrho}(x)\,dx \quad \text{for a.e. } t \geqslant 0. \tag{1.36}$$
2. *The total energy is bounded in time:*
$$\int_{\mathbf{R}} \big(\tfrac{1}{2}\varrho u^2 + U(\varrho)\big)(t,x)\,dx \leqslant \int_{\mathbf{R}} \big(\tfrac{1}{2}\bar{\varrho}\bar{u}^2 + U(\bar{\varrho})\big)(x)\,dx \quad \text{for a.e. } t \geqslant 0. \tag{1.37}$$
3. *The entropy inequalities are satisfied for all convex, subquadratic ψ:*
$$\partial_t \eta_\psi(\varrho,u) + \partial_x q_\psi(\varrho,u) \leqslant 0 \quad \text{in } \mathscr{D}'([0,\infty)\times\mathbf{R}). \tag{1.38}$$
4. *The initial data is attained in the distributional sense.*

The conservation laws (1.1) follow from (1.38) when choosing $\psi(s) = \pm 1$ for all $s \in \mathbf{R}$, or $\psi(s) = \pm s$; see table (1.21). As mentioned before, it is natural to expect that (1.38) holds for quadratic ψ as well, but we do not have a proof for this conjecture. The assumptions (1.36) and (1.37) imply that
$$\varrho \in \mathscr{L}^\infty([0,\infty), \mathscr{L}^1 \cap \mathscr{L}^\gamma(\mathbf{R})) \quad \text{and} \quad u \in \mathscr{L}^\infty([0,\infty), \mathscr{L}^2(\mathbf{R},\varrho)). \tag{1.39}$$
This implies the following bound on the momentum:
$$m := \varrho u \in \mathscr{L}^\infty([0,\infty), \mathscr{L}^1 \cap \mathscr{L}^p(\mathbf{R})), \quad p := \frac{2\gamma}{\gamma+1} > 1. \tag{1.40}$$
We can now state the main result of the paper.

Theorem 1.5 (Global Existence). *Consider the isentropic Euler equations* (1.1) *for a polytropic gas with adiabatic coefficient $\gamma > 1$. For any initial data with finite mass and total energy, there exists a finite energy solution.*

As explained above, we are not going to present an existence proof, but instead only discuss the *stability* of solutions: we consider a sequence $\{(\varrho_n, u_n)\}_n$ of finite energy solutions of (1.1) in the sense of Definition 1.4 for suitable initial data $\{(\bar{\varrho}_n, \bar{u}_n)\}_n$, and show that there exists a subsequence (still denoted by $\{(\varrho_n, u_n)\}_n$) that converges strongly towards another finite energy solution. We assume that the sequence $\{(\bar\varrho_n, \bar u_n)\}_n$ converges in a sufficiently strong sense that we can pass to the limit in nonlinear terms at initial time, and that the intial total mass and energy are bounded uniformly by constants $M > 0$ and $E > 0$ resp. The latter implies uniform boundedness of ϱ_n, u_n, and the momentum $m_n := \varrho_n u_n$ in the spaces (1.39) and (1.40).

Extracting a subsequence if necessary, we may assume that there exists a Young measure $\nu \in \mathscr{L}^\infty_{\mathrm{w}}([0,\infty) \times \mathbf{R}, \mathscr{P}([0,\infty) \times \mathbf{R}))$ such that

$$\phi(\varrho_n, u_n) \rightharpoonup \langle \phi(r,v) \rangle := \int_{[0,\infty)\times\mathbf{R}} \phi(r,v)\, \nu.(dr, dv)$$

weakly in $\mathscr{L}^1_{\mathrm{loc}}([0,\infty) \times \mathbf{R})$, for all $\phi \in \mathscr{C}([0,\infty) \times \mathbf{R})$ for which

$$\{\phi(\varrho_n, u_n)\} \text{ is locally equi-integrable in } [0,\infty) \times \mathbf{R}. \tag{1.41}$$

One can show that the condition (1.41) is satisfied for all entropies η_ψ with ψ convex and quadratic, as well as for entropy fluxes q_ψ with ψ subquadratic. This implies the existence of a *measure-valued solution* of the isentropic Euler equations: The total mass and energy are bounded in the sense that

$$\left.\begin{aligned} \int_{\mathbf{R}} \langle r \rangle(t,x)\, dx &= \int_{\mathbf{R}} \bar\varrho(x)\, dx \\ \int_{\mathbf{R}} \langle \tfrac12 r v^2 + U(v) \rangle(t,x)\, dx &\leqslant \int_{\mathbf{R}} \big(\tfrac12 \bar\varrho \bar u^2 + U(\bar\varrho)\big)(x)\, dx \end{aligned}\right. \quad \text{for a.e. } t \geqslant 0,$$

and the following entropy inequalities hold in the distributional sense:

$$\partial_t \langle \eta_\psi(r,v) \rangle + \partial_x \langle q_\psi(r,v) \rangle \leqslant 0 \tag{1.42}$$

for all ψ that are convex and subquadratic. The initial data is attained in the distributional sense. Recall that (1.42) implies the continuity and momentum equations by choosing ψ constant and linear resp.

The assumption (1.41) follows from a careful analysis of the asymptotics of $(r,v) \mapsto (\eta_\psi(r,v), q_\psi(r,v))$ and the following higher integrability:

Lemma 1.6. *There exists a constant $C > 0$ such that for all $T > 0$*

$$\sup_n \operatorname*{ess\,sup}_{y\in\mathbf{R}} \left\{ \int_{[0,T]} \Big(\varrho_n |u_n|^3 + \varrho_n^{\gamma+\theta} \Big)(t,y)\, dt \right\} \leqslant C. \tag{1.43}$$

We postpone the proof of Lemma 1.6 until Section 1.4 and refer the reader to [13] for further details on how (1.41) can be derived from (1.43).

Remark 1.7. The higher integrability result of Lemma 1.6 is sufficient to prove that the sequence of energy fluxes (corresponding to the entropy flux q_ψ with *quadratic* ψ) is uniformly bounded in $\mathscr{L}^1$ locally: We have

$$\int_K \varrho_n^\gamma |u_n|\,dx \leqslant \left(\int_K \varrho_n^{(3\gamma-1)/2}\,dx\right)^{2/3} \left(\int_K \varrho_n |u_n|^3\,dx\right)^{1/3} \tag{1.44}$$

for all $K \subset \mathbf{R}$ compact. But $(3\gamma-1)/2 = \gamma+\theta$, and so (1.44) can be bounded in space and time locally. But a local bound in $\mathscr{L}^1$ is not enough to pass to the limit – even if there are no oscillations – because of potential concentrations. Therefore we do not have a local energy inequality, only a global one.

The remaining step in the proof of the stability result is to show that the measure-valued solution just obtained is in fact a weak solution. That is, to prove that we can pass to the limit in the nonlinear terms so that

$$\langle \eta_\psi(r,v)\rangle = \eta_\psi(\varrho,u) \quad \text{and} \quad \langle q_\psi(r,v)\rangle = q_\psi(\varrho,u) \tag{1.45}$$

almost everywhere in $[0,\infty)\times\mathbf{R}$, where (ϱ,u) are defined by

$$\varrho := \langle r\rangle \quad \text{and} \quad \varrho u := \langle rv\rangle.$$

As discussed earlier, a sufficient condition for this to be true is that the Young measure $\nu_{(t,x)}$ is concentrated almost everywhere, i.e., a Dirac measure located at the position $(\varrho(t,x), u(t,x))$. To obtain this result we use the compensated compactness method in the form of the div-curl-lemma.

We again work with entropy-entropy flux pairs (η_ψ, q_ψ), but this time we consider functions ψ that may not be convex, but are smooth and compactly supported. One can show that in this case the entropy dissipation

$$\partial_t \eta_\psi(\varrho_n,u_n) + \partial_x q_\psi(\varrho_n,u_n) \text{ is a finite measure} \tag{1.46}$$

for all n. The measures in (1.46) are signed measures that depend on ψ, but their total variation can be bounded uniformly in terms of the bounds M and E on the initial mass and energy. The entropy-entropy flux pairs themselves can be controled using the following higher integrability estimate:

Lemma 1.8. *For any $T>0$ there exists a constant $C>0$ such that*

$$\sup_n \int_{[0,T]\times\mathbf{R}} \varrho_n^{\gamma+1}(t,x)\,dx\,dt \leqslant C. \tag{1.47}$$

More precisely, one can show that for any pair $\psi, \psi' \in \mathscr{D}(\mathbf{R})$ it holds

$$\left.\begin{aligned} \eta_\psi(\varrho_n,u_n) &\rightharpoonup \langle \eta_\psi(r,v)\rangle \\ q_\psi(\varrho_n,u_n) &\rightharpoonup \langle q_\psi(r,v)\rangle \end{aligned}\right\} \text{ weakly in } \mathscr{L}^{\gamma+1}_{\mathrm{loc}}([0,\infty)\times\mathbf{R}), \tag{1.48}$$

$$\left.\begin{aligned} \eta_\psi(\varrho_n,u_n)\eta_{\psi'}(\varrho_n,u_n) &\rightharpoonup \langle \eta_\psi(r,v)\rangle\langle \eta_{\psi'}(r,v)\rangle \\ \eta_\psi(\varrho_n,u_n)q_{\psi'}(\varrho_n,u_n) &\rightharpoonup \langle \eta_\psi(r,v)\rangle\langle q_\psi(r,v)\rangle \end{aligned}\right\} \text{ weakly in } \mathscr{L}^1_{\mathrm{loc}}([0,\infty)\times\mathbf{R}). \tag{1.49}$$

Again we postpone the proof of Lemma 1.8 until Section 1.4, and we refer the reader to [13] for details on how the weak convergence in (1.48) and (1.49) can be derived from (1.47). Let us just mention that the uniform bound (1.47) implies that the Young measure ν has the property that

$$\int_{[0,\infty)\times\mathbf{R}} r^{\gamma+1}\,\nu.(dr,dv) < \infty \quad \text{a.e. in } [0,\infty)\times\mathbf{R}. \tag{1.50}$$

Since the space of finite measures on $[0,\infty)\times\mathbf{R}$ is compactly embedded into the Sobolev space $\mathscr{W}^{-1,p}_{\mathrm{loc}}(\mathbf{R}^2)$ for $1 \leqslant p < 2$, and since the left-hand side of (1.46) is uniformly bounded in $\mathscr{W}^{-1,\gamma+1}_{\mathrm{loc}}(\mathbf{R}^2)$ with $\gamma+1 > 2$ because of (1.48), Murat's lemma (see [18]) implies that the sequence

$$\{\partial_t\eta_\psi(\varrho_n,u_n) + \partial_x q_\psi(\varrho_n,u_n)\}_n \text{ is precompact in } \mathscr{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^2), \tag{1.51}$$

for all $\psi \in \mathscr{D}(\mathbf{R})$. Consider now the vectors

$$G_n := \begin{pmatrix} \eta_\psi(\varrho_n,u_n) \\ q_\psi(\varrho_n,u_n) \end{pmatrix} \quad \text{and} \quad H_n := \begin{pmatrix} -q_{\psi'}(\varrho_n,u_n) \\ \eta_{\psi'}(\varrho_n,u_n) \end{pmatrix}$$

with $\psi,\psi' \in \mathscr{D}(\mathbf{R})$. Because of (1.48), the sequences $\{G_n\}_n$ and $\{H_n\}_n$ are uniformly bounded in $\mathscr{L}^2_{\mathrm{loc}}(\mathbf{R}^2)$. Moreover, because of (1.51) we have that

$$\begin{matrix} \{\operatorname{div} G_n\}_n \\ \{\operatorname{curl} H_n\}_n \end{matrix} \quad \text{are precompact in } \mathscr{H}^{-1}_{\mathrm{loc}}(\mathbf{R}^2)$$

(in $\mathbf{R}^2$ the curl is just the divergence of the rotated vector). Then the div-curl-lemma tells us that the scalar product $G_n \cdot H_n$ is weakly continuous in the distributional sense. Using (1.49), we obtain the identity

$$\langle -\eta_\psi q_{\psi'} + q_\psi \eta_{\psi'}\rangle + \langle\eta_\psi\rangle\langle q_{\psi'}\rangle - \langle q_\psi\rangle\langle\eta_{\psi'}\rangle = 0 \tag{1.52}$$

as a function in $\mathscr{L}^1_{\mathrm{loc}}(\mathbf{R}^2)$ and thus pointwise a.e. in $\mathbf{R}^2$, for all $\psi,\psi' \in \mathscr{D}(\mathbf{R})$. (We did not indicate the dependence on (r,v) to simplify the formula.) But since the entropy-entropy flux pairs are obtained by integrating corresponding kernels (see (1.20) and (1.22)) with respect to $s \in \mathbf{R}$, we actually have

$$\begin{aligned} \int_{\mathbf{R}\times\mathbf{R}} \Big(\langle -\chi(s)\sigma(s') + \sigma(s)\chi(s')\rangle + \langle\chi(s)\rangle\langle\sigma(s')\rangle - \langle\sigma(s)\rangle\langle\chi(s')\rangle \Big) \\ \times \psi(s)\psi'(s')\,ds\,ds' = 0 \quad \text{for all } \psi,\psi' \in \mathscr{D}(\mathbf{R}). \end{aligned}$$

Again this identity holds almost everyhwere in $\mathbf{R}^2$. Since the test functions ψ,ψ' are arbitrary, we finally find that for almost every $(t,x) \in \mathbf{R}^2$ the probability measure $\nu_{(t,x)}$ has the property that for a.e. $s,s' \in \mathbf{R}$

$$\langle -\chi(s)\sigma(s') + \sigma(s)\chi(s')\rangle + \langle\chi(s)\rangle\langle\sigma(s')\rangle - \langle\sigma(s)\rangle\langle\chi(s')\rangle = 0. \tag{1.53}$$

We call (1.53) the div-curl-commutator identity, and we will show in the next section that a probability measure satisfying the div-curl-commutator identity must be a Dirac measure or supported in the vacuum $\{(r,v)\colon r=0\}$. Since all entropy-entropy flux pairs $(\eta_\psi(r,v), q_\psi(r,v))$ vanish for $r=0$ for all $v \in \mathbf{R}$, we obtain (1.45), which concludes the proof of the stability result.

1.3 Young Measure Reduction

In this section, we are going to prove that any probability measure that satisfies the div-curl-commutator identity is a Dirac measure or supported in the vacuum. It will be convenient to work with the Riemann invariants $(\overline{z}, \underline{z})$ associated with (1.1), rather than with the physical variables (ϱ, u). For simplicity of notation, we will consistently denote pairs of numbers such as $(\overline{z}, \underline{z})$ by the corresponding bold symbol $\mathbf{z} := (\overline{z}, \underline{z})$. We have that

$$\overline{z}(\rho, u) = u + \varrho^\theta \quad \text{and} \quad \underline{z}(\rho, u) = u - \varrho^\theta, \tag{1.54}$$

which is equivalent to

$$\varrho(\mathbf{z}) = \left(\frac{\overline{z} - \underline{z}}{2}\right)^{1/\theta} \quad \text{and} \quad u(\mathbf{z}) = \frac{\overline{z} + \underline{z}}{2}. \tag{1.55}$$

We consider entropies and entropy fluxes as functions of $\mathbf{z}$ on

$$\mathcal{H} := H \cup V \quad \text{with} \quad \begin{aligned} H &:= \{\mathbf{z} \in \mathbf{R}^2 : \overline{z} > \underline{z}\}, \\ V &:= \{\mathbf{z} \in \mathbf{R}^2 : \overline{z} = \underline{z}\}. \end{aligned}$$

We will refer to V as the vacuum. The entropy kernels take the form

$$\begin{aligned} \chi(s|\mathbf{z}) &= \Big((\overline{z} - s)(s - \underline{z})\Big)_+^\lambda, \\ \sigma(s|\mathbf{z}) &= \left(\theta s + (1-\theta)\frac{\overline{z} + \underline{z}}{2}\right)\chi(s|\mathbf{z}) \end{aligned} \tag{1.56}$$

for all $\mathbf{z} \in \mathcal{H}$ and $s \in \mathbf{R}$ (cf. (1.19) and (1.23)). For any $\nu \in \mathscr{P}(\mathcal{H})$ let

$$\langle \phi \rangle := \int_{\mathcal{H}} \phi(\mathbf{a})\,\nu(d\mathbf{a}) \quad \text{for all } \phi \in \mathscr{C}(\mathcal{H}).$$

We say that the probability measure ν satisfies the div-curl-commutator identity if (1.53) is satisfied for a.e. $s, s' \in \mathbf{R}$. We have the following result:

Theorem 1.9 (Young Measure Reduction). *If $\nu \in \mathscr{P}(\mathcal{H})$ satisfies*

$$\int_{\mathcal{H}} W(\mathbf{a})\,\nu(d\mathbf{a}) < \infty \quad \textit{with} \quad W(\mathbf{a}) := 1 + \varrho(\mathbf{a})^{\gamma+1}$$

and the div-curl-commutator identity, then ν is either a Dirac measure located at some point in H, or the support of ν is a subset of the vacuum V.

In the following, we will only consider the case $\gamma \in (1, 5/3]$. The result can be extended to the full range of adiabatic coefficients $\gamma > 1$, by using slightly different arguments. We refer the reader to [4] for further details.

Lemma 1.10. *Let ν be the probability measure of Theorem 1.9 and define the map $s \in \mathbf{R} \mapsto \langle\chi(s)\rangle$. Then $\langle\chi\rangle \in \mathscr{C}^\alpha(\mathbf{R})$ for all $\alpha \in [0,\lambda]$, and so*

$$\mathbb{S} := \big\{ s \in \mathbf{R} \colon \langle\chi(s)\rangle > 0 \big\}$$

is open. If $\mathbb{S}$ is empty, then $\nu(H) = 0$. If $\mathbb{S}$ is nonempty, then define numbers $\underline{z} := \inf \mathbb{S}$ and $\overline{z} := \sup \mathbb{S}$ (both possibly unbounded). Then $\mathbb{S} = (\underline{z}, \overline{z})$ and

$$\operatorname{spt}\nu \cap \big\{ \mathbf{a} \in H \colon \underline{a} < \underline{z} \text{ or } \overline{z} < \overline{a} \big\} = 0. \tag{1.57}$$

Proof. Note that the function $f(t) := (1-t^2)_+^\lambda$ is bounded and Hölder continuous with Hölder exponent λ. We write the entropy kernel in the form

$$\chi(s|\mathbf{a}) = \rho(\mathbf{a})^{2\theta\lambda} f\left(\frac{s-u(\mathbf{a})}{\rho(\mathbf{a})^\theta}\right) \quad \text{for } (s,\mathbf{a}) \in \mathbf{R} \times \mathcal{H}, \tag{1.58}$$

where $\rho(\mathbf{a})$ and $u(\mathbf{a})$ are defined by (1.54). We then obtain

$$\begin{aligned} \sup_{s \neq s'} \frac{|\chi(s|\mathbf{a}) - \chi(s'|\mathbf{a})|}{|s-s'|^\alpha} &= \rho(\mathbf{a})^{(2\lambda-\alpha)\theta} \sup_{t \neq t'} \frac{|f(t)-f(t')|}{|t-t'|^\alpha} \\ &\leqslant C\rho(\mathbf{a})^{(2\lambda-\alpha)\theta}\,, \end{aligned}$$

with $C > 0$ some constant that does not depend on $\mathbf{a}$. We also have

$$\sup_{s\in\mathbf{R}} |\chi(s|\mathbf{a})| \leqslant \rho(\mathbf{a})^{2\lambda\theta}.$$

Since $0 < (2\lambda-\alpha)\theta < 1$ for all $\alpha \in [0,\lambda]$, we can now estimate

$$\begin{aligned} \sup_{s\neq s'} \frac{\big|\langle\chi(s)\rangle - \langle\chi(s')\rangle\big|}{|s-s'|^\alpha} &= \sup_{s\neq s'} |s-s'|^{-\alpha} \left| \int_{\mathcal{H}} \chi(s|\mathbf{a})\,\nu(d\mathbf{a}) - \int_{\mathcal{H}} \chi(s'|\mathbf{a})\,\nu(d\mathbf{a}) \right| \\ &\leqslant \int_{\mathcal{H}} \sup_{s\neq s'} \frac{|\chi(s|\mathbf{a}) - \chi(s'|\mathbf{a})|}{|s-s'|^\alpha}\,\nu(d\mathbf{a}) \\ &\leqslant C\int_{\mathcal{H}} W(\mathbf{a})\,\nu(d\mathbf{a}), \end{aligned}$$

which is finite by assumption on ν. The function $\langle\chi\rangle$ is bounded:

$$\begin{aligned} \sup_{s\in\mathbf{R}} |\langle\chi(s)\rangle| &= \sup_{s\in\mathbf{R}} \left| \int_{\mathcal{H}} \chi(s|\mathbf{a})\,\nu(d\mathbf{a}) \right| \\ &\leqslant \int_{\mathcal{H}} \sup_{s\in\mathbf{R}} |\chi(s|\mathbf{a})|\,\nu(d\mathbf{a}) \leqslant \int_{\mathcal{H}} W(\mathbf{a})\,\nu(d\mathbf{a}). \end{aligned}$$

This shows that $\langle\chi\rangle \in C^\alpha(\mathbf{R})$ for all $\alpha \in [0,\lambda]$, so $\mathbb{S}$ is well-defined and open.

We show next that $\mathbb{S}$ can be represented in the form

$$\mathbb{S} = \bigcup_{\mathbf{a}\in\operatorname{spt}\nu\cap H} (\underline{a}, \overline{a}). \tag{1.59}$$

Indeed assume that $\mathbf{a} \in \operatorname{spt}\nu \cap H$. Then we have $\nu(B_r(\mathbf{a}) \cap H) > 0$ for all $r > 0$, by definition of support of a measure. Therefore we obtain

$$\langle \chi(s) \rangle \geqslant \int_{B_r(\mathbf{a})} \chi(s|\mathbf{a}')\, d\nu(\mathbf{a}') > 0$$

at least for all $s \in \mathbf{R}$ with the property that $\chi(s|\mathbf{a}') > 0$ for all $\mathbf{a}' \in B_r(\mathbf{a})$. This implies $(\underline{a}+r, \overline{a}-r) \subset \mathbb{S}$. Since $r > 0$ and $\mathbf{a}$ were arbitrary, we get the $\supset$ inclusion in (1.59). For the converse direction, suppose that

$$\langle \chi(s) \rangle = \int_{\mathcal{H}} \chi(s|\mathbf{a}')\, d\nu(\mathbf{a}') > 0 \tag{1.60}$$

for some $s \in \mathbf{R}$. Since χ vanishes in the vacuum V, in (1.60) we can restrict integration to H. Then $\nu(\{\mathbf{a} \in H \colon \underline{a} < s < \overline{a}\}) > 0$, so there exists at least one point $\mathbf{a} \in \operatorname{spt}\nu$ in that set. Then $s \in (\underline{a}, \overline{a})$, and (1.59) follows. If now $\mathbb{S}$ is empty, then (1.59) implies that $\operatorname{spt}\nu \cap H = \varnothing$, thus $\nu(H) = 0$.

Let us now assume that $\mathbb{S}$ is nonempty. We define $\underline{z}, \overline{z}$ as in the statement of the lemma. Then we argue by contradiction and assume that $\mathbb{S}$ is disconnected. Since $\mathbb{S}$ is open, there exist numbers $\underline{z} < \underline{c} \leqslant \overline{c} < \overline{z}$ and $\varepsilon > 0$ such that

$$\begin{cases} \langle \chi(s) \rangle = 0 & \text{for } s \in [\underline{c}, \overline{c}], \\ \langle \chi(s) \rangle > 0 & \text{for } s \in (\underline{c}-\varepsilon, \underline{c}) \cup (\overline{c}, \overline{c}+\varepsilon). \end{cases}$$

In view of (1.59), this implies that

$$\operatorname{spt}\nu \cap \big\{\mathbf{a} \in H \colon \underline{c} < \overline{a} \text{ and } \underline{a} < \overline{c}\big\} = \varnothing. \tag{1.61}$$

Choosing $s \in (\underline{c}-\varepsilon, \underline{c})$ and $s' \in (\overline{c}, \overline{c}+\varepsilon)$ we use assumption (1.53) in the form

$$\frac{\langle -\chi(s)\sigma(s') + \sigma(s)\chi(s') \rangle}{\langle \chi(s) \rangle \langle \chi(s') \rangle} = \frac{\langle \sigma(s') \rangle}{\langle \chi(s') \rangle} - \frac{\langle \sigma(s) \rangle}{\langle \chi(s) \rangle}, \tag{1.62}$$

which is well-defined since $\langle \chi(s) \rangle \langle \chi(s') \rangle > 0$. Now note that $\chi(s|\mathbf{a})\chi(s'|\mathbf{a}) = 0$ for all $\mathbf{a} \in \operatorname{spt}\nu$, by (1.61) (see Figure 1.3). We obtain

$$-\chi(s|\mathbf{a})\sigma(s'|\mathbf{a}) + \sigma(s|\mathbf{a})\chi(s'|\mathbf{a}) = 0 \quad \text{for all } \mathbf{a} \in \operatorname{spt}\nu,$$

so the left-hand side of (1.62) vanishes. For the right-hand side we can estimate

$$\frac{\langle \sigma(s) \rangle}{\langle \chi(s) \rangle} = \theta s \frac{\langle \chi(s) \rangle}{\langle \chi(s) \rangle} + (1-\theta) \frac{\langle u\chi(s) \rangle}{\langle \chi(s) \rangle} \leqslant \theta s + (1-\theta)\underline{c} < \underline{c}.$$

Here, we have used that on the one hand

$$\operatorname{spt}\chi(s|\cdot) \cap \operatorname{spt}\nu \subset \big\{\mathbf{a} \in H \colon \overline{a} \leqslant \underline{c}\big\} \cup V \subset \big\{\mathbf{a} \in H \colon u(\mathbf{a}) \leqslant \underline{c}\big\} \cup V$$

in view of (1.61) (see again Figure 1.3) and, on the other hand, ν can not be entirely concentrated at one point where $\chi(s|\mathbf{a}) = 0$ since $\langle \chi(s) \rangle > 0$.

With the analogous estimate

$$\frac{\langle \sigma(s') \rangle}{\langle \chi(s') \rangle} = \theta s' \frac{\langle \chi(s') \rangle}{\langle \chi(s') \rangle} + (1-\theta) \frac{\langle u\chi(s') \rangle}{\langle \chi(s') \rangle} \geqslant \theta s' + (1-\theta)\overline{c} > \overline{c},$$

we obtain from (1.62) that $0 > \overline{c} - \underline{c} \geqslant 0$, which is a contradiction.

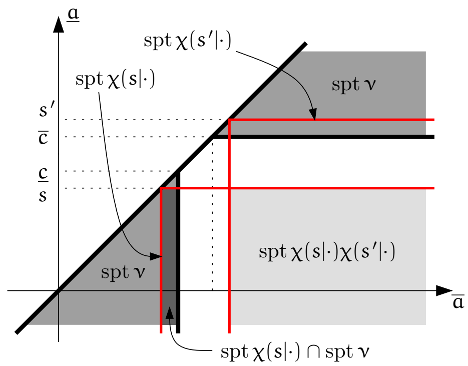

Fig. 1.2. The product $\chi(s|\cdot)\chi(s'|\cdot)$ lives outside $\operatorname{spt}\nu$.

1.3.1 Expansion of the entropy kernels

In order to show that the probability measure of Theorem 1.9 is concentrated at one point, we must understand how the entropy-entropy flux kernels behave under fractional differentiation with respect to s. For $\lambda > 0$ and suitable functions $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ we define the operators

$$\mathbf{D}f := \mathbf{F}^{-1}\big(|\cdot|^{\lambda+1}\mathbf{F}f\big), \quad \mathbf{d}f := \mathbf{F}^{-1}\big(i|\cdot|^{\lambda}\operatorname{sign}(\cdot)\mathbf{F}f\big) \tag{1.63}$$

in the distributional sense, where $\mathbf{F}$ denotes the Fourier transform. We have

$$\mathbf{D}f(s) = \frac{d}{ds}\big(\mathbf{d}f(s)\big), \tag{1.64}$$

$$\mathbf{D}\big(sf(s)\big) = s\mathbf{D}f(s) + (\lambda+1)\mathbf{d}f(s). \tag{1.65}$$

We now apply these operators to the function $f(s) := (1-s^2)_+^\lambda$ with $s \in \mathbf{R}$. According to [11], its Fourier transform is given by

$$\mathbf{F}f(z) := 2^\lambda \Gamma(\lambda+1)|z|^{-\lambda-1/2}J_{\lambda+1/2}(|z|) \tag{1.66}$$

for all $z \in \mathbf{R}$, where Γ denotes the Gamma function and $J_{\lambda+1/2}$ is the Bessel function. Despite the singular factor in (1.66), the function $\mathbf{F}f$ is bounded, due to the decaying properties of the Bessel function. We have

$$\mathbf{d}f = c\mathbf{F}^{-1}\Big(|\cdot|^{-1/2}\mathbf{F}g\Big), \tag{1.67}$$

where c is some constant and the function g is defined for all $z \in \mathbf{R}$ by

$$\mathbf{F}g(z) := i\operatorname{sign}(z)J_{\lambda+1/2}\big(|z|\big).$$

The inverse Fourier transform of $|\cdot|^{-1/2}$ induces a fractional integration operator, called Riesz potential (see [19]). Therefore (1.67) is equivalent to

$$\mathbf{d}f(s) = C|\cdot|^{-1/2} \star g(s), \quad s \in \mathbf{R}, \tag{1.68}$$

with C some new constant. Since $\mathbf{F}g$ is an odd function, we can express the inverse Fourier transform in terms of the inverse Fourier Sine transform and obtain the following explicit formula (see [12]):

$$\begin{aligned} g(s) &= \sqrt{\frac{2}{\pi}}\,\mathrm{sign}(s) \int_0^\infty J_{\lambda+1/2}(z) \sin\bigl(z|s|\bigr)\,dz \\ &= \sqrt{\frac{2}{\pi}}\,\mathrm{sign}(s) \begin{cases} \dfrac{\sin\Bigl((\lambda+\frac{1}{2})\arcsin|s|\Bigr)}{\sqrt{1-s^2}}, & |s|<1, \\[2ex] \dfrac{\cos\Bigl((\lambda+\frac{1}{2})\frac{\pi}{2}\Bigr)}{\sqrt{s^2-1}\Bigl(|s|+\sqrt{s^2-1}\Bigr)^{\lambda+1/2}}, & |s|>1. \end{cases} \end{aligned} \tag{1.69}$$

Note that g decays like $|s|^{-(\lambda+3/2)}$ as $|s| \to \infty$ and diverges like $|1-|s||^{-1/2}$ as $|s| \to 1$. This implies $g \in L^p(\mathbf{R})$ for all $p \in [1,2)$. By the Hardy-Littlewood-Sobolev theorem (see [19]), we then have $\mathbf{d}f \in L^q(\mathbf{R})$ for all $q \in (2,\infty)$. The singular behavior of $\mathbf{d}f$ and $\mathbf{D}f$ is decribed in the following proposition.

Proposition 1.11 (Fractional derivatives). *Let $f(s) = (1-s^2)_+^\lambda$ for all $s \in \mathbf{R}$, and define the fractional derivatives $\mathbf{D}f$ and $\mathbf{d}f$ by* (1.63)*. Then there exist constants A_i, $i = 1 \dots 4$, and functions r, $q \in W^{1,p}(\mathbf{R})$ for $p \in [2,\infty)$, such that in the distributional sense we have the following expansions:*

$$\begin{aligned} \mathbf{d}f(s) &= A_1\Bigl(H(s+1)+H(s-1)\Bigr) + A_2\Bigl(\mathrm{Ci}(s+1)-\mathrm{Ci}(s-1)\Bigr) + r(s), \\ \mathbf{D}f(s) &= A_1\Bigl(\delta(s+1)+\delta(s-1)\Bigr) + A_2\Bigl(\mathrm{PV}(s+1)-\mathrm{PV}(s-1)\Bigr) \\ &\quad + A_3\Bigl(H(s+1)-H(s-1)\Bigr) + A_4\Bigl(\mathrm{Ci}(s+1)+\mathrm{Ci}(s-1)\Bigr) + q(s). \end{aligned}$$

Here δ is the Dirac measure, PV *is the principal value distribution, and H denotes the Heaviside function. The function* Ci *is the Cosine integral*

$$\mathrm{Ci}(s) := -\int_{|s|}^\infty \frac{\cos t}{t}\,dt = C + \log|s| + \int_0^{|s|} \frac{\cos t - 1}{t}\,dt, \quad s \in \mathbf{R}, \tag{1.70}$$

with $C > 0$ some constant. For simplicity, we will treat the distributions δ and PV *as if they were functions. The coefficients A_1 and A_2 are not both equal to zero. Moreover, if $\gamma = (M+2)/M$ with $M \in \mathbf{N}$ odd, then $A_2 = A_4 = 0$.*

Remark 1.12. By Sobolev embedding, the remainders are Hölder continuous: We have $r, q \in C^\alpha(\mathbf{R})$ for all exponents $\alpha \in [0,1)$. In particular, the functions are bounded. Moreover, we get $r, q \in W^{1,p}_{\mathrm{loc}}(\mathbf{R})$ for all $p \in [1,\infty)$.

We refer the reader to [13] for a proof of Proposition 1.11. It is used to find expansions for the entropy kernel. Note that

$$\chi(s|\mathbf{a}) = \rho(\mathbf{a})^{2\theta\lambda}\, f\bigg(\frac{s-u(\mathbf{a})}{\rho(\mathbf{a})^\theta}\bigg), \quad (s,\mathbf{a}) \in \mathbf{R}\times\mathcal{H}.$$

Therefore the chain rule implies the identities

$$\begin{aligned}
&\mathbf{d}\chi(s|\mathbf{a}) \\
&\quad = \rho(\mathbf{a})^{\theta\lambda}\bigg(A_1\Big(H(s-\underline{a})+H(s-\overline{a})\Big)+A_2\Big(\mathrm{Ci}(s-\underline{a})-\mathrm{Ci}(s-\overline{a})\Big)\bigg) \\
&\quad + \rho(\mathbf{a})^{\theta\lambda}\, r\bigg(\frac{s-u(\mathbf{a})}{\rho(\mathbf{a})^\theta}\bigg),
\end{aligned} \tag{1.71}$$

$$\begin{aligned}
&\mathbf{D}\chi(s|\mathbf{a}) \\
&\quad = \rho(\mathbf{a})^{\theta\lambda}\bigg(A_1\Big(\delta(s-\underline{a})+\delta(s-\overline{a})\Big)+A_2\Big(\mathrm{PV}(s-\underline{a})-\mathrm{PV}(s-\overline{a})\Big)\bigg) \\
&\quad + \rho(\mathbf{a})^{\theta(\lambda-1)}\bigg(A_3\Big(H(s-\underline{a})-H(s-\overline{a})\Big)+A_4\Big(\mathrm{Ci}(s-\underline{a})+\mathrm{Ci}(s-\overline{a})\Big)\bigg) \\
&\quad + \rho(\mathbf{a})^{\theta(\lambda-1)}\bigg(-A_4 2\theta\log\rho(\mathbf{a})+q\bigg(\frac{s-u(\mathbf{a})}{\rho(\mathbf{a})^\theta}\bigg)\bigg)
\end{aligned} \tag{1.72}$$

in the distributional sense in s for all $\mathbf{a}\in\mathcal{H}$. Using (1.56) and the product rule (1.65) we obtain similar identities for the entropy-flux kernel σ. For $\gamma = 5/3$ we have $A_2 = A_4 = 0$, so (1.71) and (1.72) do not contain PV and Ci.

1.3.2 Proof of the reduction result

We choose nonnegative test functions $\varphi, \varphi' \in \mathscr{D}(\mathbf{R})$ with support in the interval $[-1,1]$ and with integral equal to one. For $\varepsilon > 0$ we put

$$\varphi_\varepsilon(s) := \varepsilon^{-1}\varphi(s/\varepsilon), \quad \varphi'_\varepsilon(s) := \varepsilon^{-1}\varphi'(s/\varepsilon)$$

for all $(s,\varepsilon)\in\mathbf{R}\times(0,1)$. We then mollify the entropy kernels: Let

$$\chi_\varepsilon(s|\mathbf{a}) := \chi(\cdot|\mathbf{a})\star\varphi_\varepsilon(s), \quad \sigma_\varepsilon(s|\mathbf{a}) := \sigma(\cdot|\mathbf{a})\star\varphi_\varepsilon(s)$$

for all $(s,\mathbf{a})\in\mathbf{R}\times\mathcal{H}$, and define $(\chi'_\varepsilon, \sigma'_\varepsilon)$ analogously, using the mollifier φ'_ε instead. We assume that φ and φ' are chosen in such a way that

$$Z := \iint_{\mathbf{R}\times\mathbf{R}} H(t-s)\Big(\varphi(t)\varphi'(s)-\varphi(s)\varphi'(t)\Big)\,ds\,dt \tag{1.73}$$

is a positive number. As shown in [3], this is always possible.

The proof of Theorem 1.9 relies on the following two propositions.

Proposition 1.13. *There exists a constant $B > 0$ depending only on λ and Z defined in* (1.73) *such that for any nonnegative $\zeta \in \mathscr{D}(\mathbf{R})$ we have*

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}} \Big\langle \mathbf{D}\chi_\varepsilon(t)\mathbf{D}\sigma'_\varepsilon(t) - \mathbf{D}\sigma_\varepsilon(t)\mathbf{D}\chi'_\varepsilon(t)\Big\rangle \big\langle\chi(t)\big\rangle\zeta(t)\,dt$$
$$= B\int_{\mathcal{H}} \rho(\mathbf{a})^{1-\theta}\Big(\big\langle\chi(\overline{a})\big\rangle\zeta(\overline{a}) + \big\langle\chi(\underline{a})\big\rangle\zeta(\underline{a})\Big)\,\nu(d\mathbf{a}).$$

Proposition 1.14. *For any test function $\zeta \in \mathscr{D}(\mathbf{R})$ we have*

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}} \Big\langle \chi(t)\mathbf{D}\sigma'_\varepsilon(t) - \sigma(t)\mathbf{D}\chi'_\varepsilon(t)\Big\rangle \big\langle\mathbf{D}\chi_\varepsilon(t)\big\rangle\zeta(t)\,dt$$
$$= \lim_{\varepsilon\to 0}\int_{\mathbf{R}} \Big\langle \chi(t)\mathbf{D}\sigma_\varepsilon(t) - \sigma(t)\mathbf{D}\chi_\varepsilon(t)\Big\rangle \big\langle\mathbf{D}\chi'_\varepsilon(t)\big\rangle\zeta(t)\,dt.$$

We refer the reader to [13] for proofs of Propositions 1.13 and 1.14. Let us show how they imply Theorem 1.9. We multiply (1.53) by $\langle\chi(t)\rangle$ and get

$$\Big\langle\chi(s)\sigma(s') - \sigma(s)\chi(s')\Big\rangle\big\langle\chi(t)\big\rangle = \Big(\big\langle\chi(s)\big\rangle\big\langle\sigma(s')\big\rangle - \big\langle\sigma(s)\big\rangle\big\langle\chi(s')\big\rangle\Big)\big\langle\chi(t)\big\rangle$$

for almost all $(s, s', t) \in \mathbf{R}^3$. Cyclic permutation of the variables yields

$$\Big\langle\chi(s')\sigma(t) - \sigma(s')\chi(t)\Big\rangle\big\langle\chi(s)\big\rangle = \Big(\big\langle\chi(s')\big\rangle\big\langle\sigma(t)\big\rangle - \big\langle\sigma(s')\big\rangle\big\langle\chi(t)\big\rangle\Big)\big\langle\chi(s)\big\rangle,$$
$$\Big\langle\chi(t)\sigma(s) - \sigma(t)\chi(s)\Big\rangle\big\langle\chi(s')\big\rangle = \Big(\big\langle\chi(t)\big\rangle\big\langle\sigma(s)\big\rangle - \big\langle\sigma(t)\big\rangle\big\langle\chi(s)\big\rangle\Big)\big\langle\chi(s')\big\rangle.$$

Summing up all terms, the right-hand sides cancel out, and we find

$$\Big\langle\chi(s)\sigma(s') - \sigma(s)\chi(s')\Big\rangle\big\langle\chi(t)\big\rangle$$
$$= \Big\langle\chi(t)\sigma(s') - \sigma(t)\chi(s')\Big\rangle\big\langle\chi(s)\big\rangle - \Big\langle\chi(t)\sigma(s) - \sigma(t)\chi(s)\Big\rangle\big\langle\chi(s')\big\rangle.$$

We apply the fractional differentiation operator $\mathbf{D}$ with respect to s and s', and then integrate against the mollifiers $\varphi_\varepsilon(t-s)$ and $\varphi'_\varepsilon(t-s')$ as defined above. Finally, we multiply the resulting terms by some nonnegative test function $\zeta \in \mathscr{D}(\mathbf{R})$ and integrate in t over $\mathbf{R}$. Then

$$\int_{\mathbf{R}} \Big\langle \mathbf{D}\chi_\varepsilon(t)\mathbf{D}\sigma'_\varepsilon(t) - \mathbf{D}\sigma_\varepsilon(t)\mathbf{D}\chi'_\varepsilon(t)\Big\rangle \big\langle\chi(t)\big\rangle\zeta(t)\,dt$$
$$= \int_{\mathbf{R}} \Big\langle \chi(t)\mathbf{D}\sigma'_\varepsilon(t) - \sigma(t)\mathbf{D}\chi'_\varepsilon(t)\Big\rangle \big\langle\mathbf{D}\chi_\varepsilon(t)\big\rangle\zeta(t)\,dt$$
$$- \int_{\mathbf{R}} \Big\langle \chi(t)\mathbf{D}\sigma_\varepsilon(t) - \sigma(t)\mathbf{D}\chi_\varepsilon(t)\Big\rangle \big\langle\mathbf{D}\chi'_\varepsilon(t)\big\rangle\zeta(t)\,dt.$$

According to Proposition 1.13, the right-hand side converges to zero as $\varepsilon \to 0$ since the two terms have the same limit. Proposition 1.14 describes the limit of the left-hand side. Sending $\varepsilon \to 0$, we arrive at the identity

$$B \int_{\mathcal{H}} \rho(\mathbf{a})^{1-\theta} \Big(\big\langle \chi(\overline{a}) \big\rangle \zeta(\overline{a}) + \big\langle \chi(\underline{a}) \big\rangle \zeta(\underline{a}) \Big) \, \nu(d\mathbf{a}) = 0. \tag{1.74}$$

The integrand in (1.74) is nonnegative. Choosing a monotone sequence of test functions $\zeta_k \in \mathscr{D}(\mathbf{R})$ with $0 \leqslant \zeta_k \leqslant 1$ and $\zeta_k \longrightarrow 1$ as $k \to \infty$, we get

$$\int_{\mathcal{H}} \rho(\mathbf{a})^{1-\theta} \langle \chi(\overline{a}) \rangle \, \nu(d\mathbf{a}) = 0 \quad \text{and} \quad \int_{\mathcal{H}} \rho(\mathbf{a})^{1-\theta} \langle \chi(\underline{a}) \rangle \, \nu(d\mathbf{a}) = 0, \tag{1.75}$$

by monotone convergence. Recall that the constant B is strictly positive.

Consider the interval $\mathbb{S} = (\underline{z}, \overline{z})$ defined in Lemma 1.10. If $\mathbb{S} = \varnothing$, then the representation (1.59) implies that $\operatorname{spt} \nu \subset V$. If $\mathbb{S} \neq \varnothing$, then we find

$$\operatorname{spt} \nu \cap \big\{ \mathbf{a} \in H \colon \overline{a} > \overline{z} \text{ or } \underline{a} < \underline{z} \big\} = \varnothing,$$

see Figure 1.3. Since $\langle \chi(s) \rangle > 0$ for all $s \in \mathbb{S}$, from (1.75) and (1.59) we get

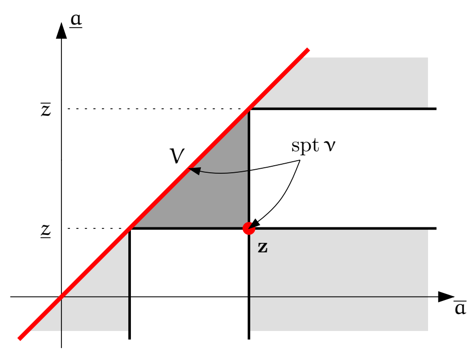

Fig. 1.3. The $\operatorname{spt} \nu$ is either the point $\mathbf{z}$ or the vacuum V.

$$\operatorname{spt} \nu \cap \big\{ \mathbf{a} \in H \colon \underline{z} < \underline{a} < \overline{z} \big\} = \varnothing \quad \text{and} \quad \operatorname{spt} \nu \cap \big\{ \mathbf{a} \in H \colon \underline{z} < \overline{a} < \overline{z} \big\} = \varnothing;$$

see again Figure 1.3. Therefore the measure ν must be contained in the vacuum V and in the isolated point $\mathbf{z} := (\underline{z}, \overline{z}) \in H$. We make an ansatz

$$\nu = (1 - \omega) \nu_V + \omega \delta_{\mathbf{z}} \quad \text{for some } \omega \in [0, 1],$$

with ν_V a probability measure supported in the vacuum V. Using this measure in the commutator relation (1.53), we find the identity

$$(\omega - \omega^2) \Big(- \chi(s|\mathbf{z}) \sigma(s'|\mathbf{z}) + \sigma(s|\mathbf{z}) \chi(s'|\mathbf{z}) \Big) = 0 \quad \text{for a.e. } s, s' \in \mathbf{R}.$$

For some $s, s' \in \mathbb{S}$ with $s \neq s'$ the second factor does not vanish, which implies that $\omega \in \{0, 1\}$. If $\omega = 0$, then ν is supported in the vacuum V. If $\omega = 1$, then ν is a Dirac measure at the point $\mathbf{z}$. This proves Theorem 1.9.

1.4 Higher Integrability

In this section, we will sketch the proofs of the two higher integrability results Lemma 1.6 and 1.8. The first result follows from the entropy inequality (1.38) for a suitable choice of *nonconvex* weight function ψ.

Proof (of Lemma 1.6). We only sketch the argument and refer the reader to [13] for more details. We choose $\psi(s) := \frac{1}{2}s|s|$ for all $s \in \mathbf{R}$ (which is nonconvex) and consider the corresponding entropy-entropy flux pairs (η_ψ, q_ψ). One can show that the following holds in the distributional sense:

$$\partial_t \eta_\psi(\varrho_n, u_n) + \partial_x q_\psi(\varrho_n, u_n) =: \mu_n \tag{1.76}$$

for every n, with μ_n a signed measure on $[0,\infty) \times \mathbf{R}$ whose total variation can be bounded uniformly in terms of the bounds M and E on the initial total mass and energy. We now integrate (1.76) against $\mathbf{1}_{[0,T]\times[y,\infty)}(t,x)$. Applying a standard approximation argument, we obtain that

$$\begin{aligned}\int_{[0,T]} & q_\psi(\varrho_n, u_n)(t,y)\,dt \\ &= \int_{[y,\infty)} \eta_\psi(\varrho_n, u_n)(T,x)\,dx - \int_{[y,\infty)} \eta_\psi(\varrho_n, u_n)(0,x)\,dx \\ &\quad + \int_{[0,T]\times[y,\infty)} \mu_n(dx,dt)\end{aligned} \tag{1.77}$$

for a.e. $T \in [0,\infty)$ and $y \in \mathbf{R}$. We used that the entropy flux $q_\psi(\varrho_n, u_n)$ decays at infinity for any $t \geqslant 0$ (because of the bound (1.48)). Now

$$\sup_n \left| \int_{[0,T]\times[y,\infty)} \mu_n(dx,dt) \right| \leqslant C,$$

with $C > 0$ some constant depending on M and E only. Moreover, we have

$$\sup_n \left| \int_{[y,\infty)} \eta_\psi(\varrho_n, u_n)(t,x)\,dx \right| \leqslant \sup_n \int_{\mathbf{R}} \big(\tfrac{1}{2}\varrho_n u_n^2 + U(\varrho_n)\big)(t,x)\,dx$$

for $t \in \{0, T\}$, which is bounded by E. We used that the total energy is the second s-moment of the entropy kernel χ, and therefore dominates η_ψ with ψ as above. Finally, we use the fact that there exists $\delta > 0$ such that

$$q_\psi(r,v) \geqslant \delta\Big(r|v|^3 + r^{\gamma+\theta}\Big) \quad \text{for all } (r,v) \in [0,\infty) \times \mathbf{R}.$$

We refer the reader to [16] for a proof. Then the result follows.

Lemma 1.8 follows from a commutator estimate.

Proof (of Lemma 1.8). For every n there exists $h_n\colon [0,\infty)\times \mathbf{R} \longrightarrow \mathbf{R}$ with

$$\partial_t h_n = -\varrho_n u_n \quad \text{and} \quad \partial_x h_n = \varrho_n \tag{1.78}$$

in the distributional sense, and for which

$$0 \leqslant h_n(t,x) \leqslant M \quad \text{for a.e. } (t,x). \tag{1.79}$$

Indeed we observe first that a function h_n satisfying (1.78) always exists since the continuity equation (which is satisfied by (ϱ_n, u_n)) implies that the mixed second derivatives of h_n commute. We notice that for almost every $t \geqslant 0$, the map $x \mapsto h_n(t,x)$ is absolutely continuous and nondecreasing. Since the total mass is preserved we conclude that for a.e. $t \geqslant 0$ we have the identity

$$\lim_{x\to\infty} h_n(t,x) - \lim_{x\to-\infty} h_n(t,x) = M. \tag{1.80}$$

On the other hand, since for any t the functions $(\varrho_n u_n)(t,\cdot)$ decay at infinity because of (1.40), the first identity in (1.78) implies that

$$\lim_{x\to-\infty} h_n(t,x) = \lim_{x\to-\infty} h_n(0,x)$$

for a.e. $t \geqslant 0$. Normalizing h_n so that $\lim_{x\to-\infty} h_n(0,x) = 0$, we get (1.79).

Step 1. We will prove that h_n is locally Hölder continuous in both variables, with constants that are bounded uniformly in n. The Hölder continuity of h_n in space follows easily from (1.37) and (1.78): Let $K \subset \mathbf{R}$ be a compact subset. For all points $x_1, x_2 \in K$ we can then estimate

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geqslant 0} |h_n(t,x_2) - h_n(t,x_1)| \\
&\quad \leqslant \operatorname*{ess\,sup}_{t\geqslant 0} \int_{x_1}^{x_2} \varrho_n(t,x)\,dx \\
&\quad \leqslant |x_2 - x_1|^{(\gamma-1)/\gamma} \operatorname*{ess\,sup}_{t\geqslant 0} \left(\int_{x_1}^{x_2} \varrho_n^\gamma(t,x)\,dx \right)^{1/\gamma}.
\end{aligned}$$

The second factor can be estimated by (1.37). We find

$$\operatorname*{ess\,sup}_{t\geqslant 0} |h_n(t,x_2) - h_n(t,x_1)| \leqslant C_1 |x_2 - x_1|^{(\gamma-1)/\gamma}, \tag{1.81}$$

with $C_1 > 0$ some constant depending on E.

To prove the Hölder continuity in time we first fix a mollifier φ_δ with the standard properties $\varphi_\delta \geqslant 0$, $\int \varphi_\delta\,dx = 1$, and $\operatorname{spt} \varphi_\delta \subset (-\delta, \delta)$. The parameter $\delta > 0$ will be chosen later on. We obtain from (1.81) that for all $x \in K$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geqslant 0} \left| \left(\int_{\mathbf{R}} \varphi_\delta(x-y) h_n(t,y)\,dy \right) - h_n(t,x) \right| \\
&\quad \leqslant C_1 \int_{\mathbf{R}} \varphi_\delta(x-y) |x-y|^{(\gamma-1)/\gamma}\,dy \\
&\quad \leqslant C_1 \delta^{(\gamma-1)/\gamma}.
\end{aligned}$$

For any $t_1, t_2 \geqslant 0$ and $x \in \mathbf{R}$ we therefore get that

$$\begin{aligned}
&|h_n(t_2,x) - h_n(t_1,x)| \\
&\quad \leqslant 2C_1\delta^{(\gamma-1)/\gamma} + \left| \int_{\mathbf{R}} \varphi_\delta(x-y) \big(h_n(t_2,y) - h_n(t_1,y)\big)\, dy \right| \\
&\quad = 2C_1\delta^{(\gamma-1)/\gamma} + \left| \int_{t_1}^{t_2} \int_{\mathbf{R}} \varphi_\delta(x-y)\, (\varrho_n u_n)(t,y)\, dy\, dt \right|. \qquad (1.82)
\end{aligned}$$

Now note that the energy bound (1.37) implies the estimate

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geqslant 0} \int_{\mathbf{R}} |(\varrho_n u_n)(t,x)|^{2\gamma/(\gamma+1)}\, dx \\
&\quad \leqslant \operatorname*{ess\,sup}_{t\geqslant 0} \left(\int_{\mathbf{R}} \varrho_n^\gamma(t,x)\, dx \right)^{1/(\gamma+1)} \left(\int_{\mathbf{R}} (\varrho_n u_n^2)(t,x)\, dx \right)^{\gamma/(\gamma+1)} \\
&\quad \leqslant C_2, \qquad (1.83)
\end{aligned}$$

with $C_2 > 0$ some constant depending on E. Using this in (1.82) and optimizing in δ, we arrive at the following estimate: for any $t_1, t_2 \geqslant 0$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{x\in\mathbf{R}} |h_n(t_2,x) - h_n(t_1,x)| \\
&\quad \leqslant 2C_1\delta^{(\gamma-1)/\gamma} + C_2^{(\gamma+1)/2\gamma} \|\varphi\|_{L^\infty(\mathbf{R})} \delta^{-(\gamma+1)/2\gamma} |t_1 - t_2| \\
&\quad \leqslant C_3 |t_1 - t_2|^{2(\gamma-1)/(3\gamma-1)}
\end{aligned}$$

for some constant $C_3 > 0$. This establishes the first part of the lemma.

Step 2. Let φ_ε be a standard mollifier in $\mathbf{R}^2$ and, after extending h_n by zero to all of $\mathbf{R}^2$, define the smooth function $h_{n,\varepsilon} := h_n \star \varphi_\varepsilon$. The following identity is true in the distributional sense in $[0,\infty) \times \mathbf{R}$:

$$\begin{aligned}
&\partial_t\Big(\varrho_n u_n\, h_{n,\varepsilon}\Big) + \partial_x\Big(\big(\varrho_n u_n^2 + P(\varrho_n)\big)\, h_{n,\varepsilon}\Big) \\
&\quad = \Big\{ \partial_t(\varrho_n u_n) + \partial_x\big(\varrho_n u_n^2 + P(\varrho_n)\big) \Big\}\, h_{n,\varepsilon} \\
&\quad + \Big\{ \varrho_n u_n\, (\partial_t h_{n,\varepsilon}) + \big(\varrho_n u_n^2 + P(\varrho_n)\big)\, (\partial_x h_{n,\varepsilon}) \Big\}.
\end{aligned}$$

The first term on the right-hand side vanishes because of the momentum equation (which is satisfied by (ϱ_n, u_n)). Moreover, we have $h_{n,\varepsilon} \longrightarrow h_n$ uniformly on compact sets as $\varepsilon \to 0$, by the Hölder continuity of h_n just established.

We have $\partial_t h_{n,\varepsilon} \longrightarrow \partial_t h_n$ and $\partial_x h_{n,\varepsilon} \longrightarrow \partial_x h_n$ in $\mathscr{L}^1_{\mathrm{loc}}([0,\infty) \times \mathbf{R})$. By boundedness of (ϱ_n, u_n) and (1.78), we find that in distributional sense

$$P(\varrho_n)\varrho_n = \partial_t\Big(\varrho_n u_n\, h_n\Big) + \partial_x\Big(\big(\varrho_n u_n^2 + P(\varrho_n)\big)\, h_n\Big) \qquad (1.84)$$

We test (1.84) against a monotone sequence of functions $\zeta_k \in \mathscr{D}([0,\infty) \times \mathbf{R})$ with $0 \leqslant \zeta_k \leqslant 1$ and $\zeta_k \to \mathbf{1}_{[0,T]\times\mathbf{R}}$ for some $T > 0$. We have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\geqslant 0} \int_{\mathbf{R}} |(\varrho_n u_n)(t,x)|\,dx \\
&\quad\leqslant \operatorname*{ess\,sup}_{t\geqslant 0} \bigg(\int_{\mathbf{R}} \varrho_n(t,x)\,dx \bigg)^{1/2} \bigg(\int_{\mathbf{R}} (\varrho_n u_n^2)(t,x)\,dx \bigg)^{1/2},
\end{aligned}$$

which can be estimated against $\sqrt{2ME}$. Since the total energy vanishes at infinity for any $t \geqslant 0$ (see (1.79)), the last term in (1.84) does not contribute. Since $h_n \geqslant 0$ is uniformly bounded by M, we obtain that for all n

$$\int_{[0,T]\times\mathbf{R}} \varrho_n^{\gamma+1}(t,x)\,dx\,dt \leqslant 2M\sqrt{2ME}. \tag{1.85}$$

This concludes the proof of the lemma.

1.5 Navier-Stokes Equations

As mentioned in the introduction, there are various ways to generate sequences of approximate solutions to the isentropic Euler equations that could be used to established global existence of solutions to (1.1) by proving that a suitable subsequence converges strongly. One method that is very interesting from the physical point of view is to start from solutions of the Navier-Stokes equations: One considers a sequence $\{(\varrho_\varepsilon, u_\varepsilon)\}_\varepsilon$ of solutions of

$$\left.\begin{aligned}
\partial_t \varrho_\varepsilon + \partial_x(\varrho_\varepsilon u_\varepsilon) &= 0 \\
\partial_t(\varrho_\varepsilon u_\varepsilon) + \partial_x\big(\varrho_\varepsilon u_\varepsilon^2 + P(\varrho_\varepsilon)\big) &= \varepsilon \partial_{xx} u_\varepsilon
\end{aligned}\right\} \quad \text{in } [0,\infty)\times\mathbf{R}, \tag{1.86}$$

where $\varepsilon > 0$ represents the physical viscosity. The long-standing conjecture has been that in the limit of vanishing viscosity $\varepsilon \to 0$, the solutions of (1.86) converge to solutions of the isentropic Euler equations (1.1). A mathematical proof of this conjecture, however, has been found only recently by Chen and Perepelitsa; see [4]. Their argument largely follows the strategy outlined in the previous sections, but there is one very interesting new ingredient: a new bound on the derivative of the density ϱ_ε. Let us briefly sketch this result and show the connections to the framework above.

We first remark that – similar to the case of the isentropic Euler equations with additional geometric effects – it seems not possible to establish uniform $\mathscr{L}^\infty$-bounds on solutions $(\varrho_\varepsilon, u_\varepsilon)$ of the Navier-Stokes equations (1.86). The natural bounds of finite total mass and total energy, however, again apply without any difficulties. One can show that solutions $(\varrho_\varepsilon, u_\varepsilon)$ of (1.86) exist for suitable initial data $(\bar\varrho, \bar u)$ and satisfy an energy inequality like

$$\int_{\mathbf{R}} \big(\tfrac{1}{2}\varrho_\varepsilon u_\varepsilon^2 + U(\varrho_\varepsilon)\big)(T,x)\,dx + \varepsilon\int_0^T\int_{\mathbf{R}} (\partial_x u_\varepsilon)^2(t,x)\,dx\,dt \\ \leqslant \int_{\mathbf{R}} \big(\tfrac{1}{2}\bar\varrho\bar u^2 + U(\bar\varrho)\big)(x)\,dx < \infty. \qquad (1.87)$$

(The actual energy inequality in [4] is more complicated than the one in (1.87) since the authors consider the perturbation around a fixed background profile with strictly positive density everywhere in $\mathbf{R}$, which obviously cannot have finite mass. We skip some details to simplify the presentation.) Let us mention in passing that proving existence of solutions to (1.86) is not trivial at all since the parabolic regularization affects only the velocity, and not the density.

The uniform bounds implied by (1.87) yield weak* precompactness of the sequence $\{(\varrho_\varepsilon, u_\varepsilon)\}_\varepsilon$ and convergence (along a subsequence) towards a suitable measure-valued solution of (1.1). To obtain strong convergence, the assumptions required for the compensated compactness method need to be checked: for all smooth and compactly support ψ the entropy dissipation

$$\partial_t\eta_\psi(\varrho_\varepsilon, u_\varepsilon) + \partial_x q_\psi(\varrho_\varepsilon, u_\varepsilon) \qquad (1.88)$$

must belong to a strongly precompact subset in $\mathscr{H}^{-1}_{\mathrm{loc}}([0,\infty)\times\mathbf{R})$. In slight abuse of notation, we may consider the entropy-entropy flux pairs (η_ψ, q_ψ) as functions of the density ϱ_ε and the momemtum $m_\varepsilon := \varrho_\varepsilon u_\varepsilon$. We find

$$\begin{aligned}\partial_t\eta_\psi(\varrho_\varepsilon, u_\varepsilon) + \partial_x q_\psi(\varrho_\varepsilon, u_\varepsilon) &= \partial_x\Big(\sqrt{\varepsilon}\big(\partial_m\eta_\psi(\varrho_\varepsilon, m_\varepsilon)\sqrt{\varepsilon}\partial_x u_\varepsilon\big)\Big)\\ &\quad - \partial_{um}\eta_\psi(\varrho_\varepsilon, m_\varepsilon)\,\varepsilon|\partial_x u_\varepsilon|^2\\ &\quad - \partial_{\varrho m}\eta_\psi(\varrho_\varepsilon, m_\varepsilon)\sqrt{\varepsilon}\partial_x\varrho_\varepsilon\sqrt{\varepsilon}\partial_x u_\varepsilon\end{aligned}$$

in the distributional sense. Because of the inequality (1.87), the term $\sqrt{\varepsilon}\partial_x u_\varepsilon$ is uniformly bounded in $\mathscr{L}^2([0,T]\times\mathbf{R})$ for all $T>0$. The new estimate in [4] is that also the density is more regular. More precisely, the term $\sqrt{\varepsilon}\varrho_\varepsilon^{\theta-1}(\partial_x\varrho_\varepsilon)$ is uniformly bounded in $\mathscr{L}^2([0,T]\times\mathbf{R})$. One can then show that

$$\begin{aligned}&\big\|\partial_{um}\eta_\psi(\varrho_\varepsilon, m_\varepsilon)\,\varepsilon|\partial_x u_\varepsilon|^2 + \partial_{\varrho m}\eta_\psi(\varrho_\varepsilon, m_\varepsilon)\sqrt{\varepsilon}\partial_x\varrho_\varepsilon\sqrt{\varepsilon}\partial_x u_\varepsilon\big\|_{\mathscr{L}^1([0,T]\times\mathbf{R})}\\ &\quad\leqslant C\big\|\big(\sqrt{\varepsilon}\partial_x u_\varepsilon, \sqrt{\varepsilon}\varrho_\varepsilon^{\theta-1}(\partial_x\varrho_\varepsilon)\big)\big\|_{\mathscr{L}^2([0,T]\times\mathbf{R})},\end{aligned}$$

which is uniformly bounded, for all smooth and compactly supported ψ. Here $C>0$ is some constant that depends only on ψ. Similarly, one has

$$\big\|\sqrt{\varepsilon}\big(\partial_m\eta_\psi(\varrho_\varepsilon, m_\varepsilon)\sqrt{\varepsilon}\partial_x u_\varepsilon\big)\big\|_{\mathscr{L}^2([0,T]\times\mathbf{R})} \leqslant C\sqrt{\varepsilon}\big\|\sqrt{\varepsilon}\partial_x u_\varepsilon\big\|_{\mathscr{L}^2([0,T]\times\mathbf{R})}$$

for a suitable constant $C>0$ that only depends on ψ. The right-hand side converges to zero as $\varepsilon\to 0$. Then Murat's lemma (see [18]) implies the strong precompactness of the entropy dissipation (1.88). By repeating the arguments of the previous sections, one can show that the Young measure found earlier is concentrated a.e., and so the the measure-valued solution is actually a weak solution of the isentropic Euler equations; see [4] for more details.

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