CONVERGENCE OF THIN FILM APPROXIMATION FOR A SCALAR CONSERVATION LAW

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Abstract. In this paper we consider the thin film approximation of a one-d scalar conservation law with strictly convex flux. We prove that the sequence of approximate solutions converges to the unique Kružkov solution.

1. Introduction

The vanishing viscosity method plays a prominent role in the theory of scalar conservation laws: To study the properties of solutions of the equation, one first adds a small amount of viscosity and considers

$$u_t^\varepsilon + f(u^\varepsilon)_x \;=\; \varepsilon u_{xx}^\varepsilon \quad \text{for } \varepsilon > 0\,, \tag{1.1}$$

and then sends ε to zero. If the limit $u \equiv \lim_{\varepsilon\to 0} u^\varepsilon$ exists, and if the convergence is strong enough to give sense to the nonlinearity $f(u)$, then u is a solution of $u_t + f(u)_x = 0$. It is well-known that solutions produced by this method are exactly the entropy solutions of the scalar conservation law, which are characterized by a family of inequalities: for all convex entropy–entropy flux pairs (η, q)

$$\eta(u)_t + q(u)_x \;\leq\; 0 \quad \text{in } \mathcal{D}'\,. \tag{1.2}$$

For bounded initial data, entropy solutions exist and are unique, see Kružkov [7].

In this paper, we consider a different approximation for a particular flux function $f(u)$. Instead of using the second order nondegenerate differential operator $\varepsilon u_{xx}^\varepsilon$, we use a fourth order degenerate one. This operator is intensely studied in the theory of thin films moving on a flat surface, where it is derived from the incompressible Navier-Stokes equation under the assumption of small film height and a *slip condition* on the substrate, see [1]. We will consider the problem

$$u_t^\varepsilon + \Big\{(u^\varepsilon)^n(1 + \varepsilon u_{xxx}^\varepsilon)\Big\}_x \;=\; 0 \quad \text{for } \varepsilon > 0\,, \tag{1.3}$$

and then study the limit $\varepsilon \to 0$. Equation (1.3) models the dynamics of a thin film of fluid on a vertical plane under the influence of gravity, see [1] and [5]. It

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has the property that the film height u^ε stays nonnegative for positive times if the initial data is. Although physical reasoning allows n in the range $(1,3]$, in this paper we only consider $n \in [\frac{3}{2}, 2)$. Using similar arguments, it is in fact possible to extend the result to $n \in (1, \frac{3}{2})$, see Remark 2.3, but we do not know how to handle the case $n \in [2,3]$ which poses additional difficulties. We will prove that, as $\varepsilon \to 0$, the functions u^ε converge to the entropy solution of the underlying scalar conservation law. This is in accordance with numerical experiments in [1] which also suggest that for *nonconvex* flux (modelling a film under the influence of gravity and Marangoni stress) entropy violating shocks may occur. Higher-order viscous approximations of scalar conservation laws have been studied in [9].

Compared to (1.1), the approximation (1.3) has a number of peculiarities, which makes the problem nontrivial. The main obstacle is the lack of entropies. Recall that for (1.1) any convex function η may serve as an entropy, since we can multiply the equation by $\eta'(u^\varepsilon)$ and then transform the right-hand side into

$$\varepsilon\eta'(u^\varepsilon)u^\varepsilon_{xx} \;=\; \varepsilon\eta(u^\varepsilon)_{xx} - \varepsilon\eta''(u^\varepsilon)\big(u^\varepsilon_x\big)^2.$$

In distributional sense, this quantity converges to a nonpositive Radon measure as $\varepsilon \to 0$, thereby yielding the entropy inequality (1.2). For (1.3), this is no longer true. In fact, only the one-parameter family

$$\eta_p(u) \equiv \tfrac{1}{p}u^p \quad \text{for } p \in [2-n, 3-n] \tag{1.4}$$

gives a right-hand side which in the limit $\varepsilon \to 0$ converges to something nonpositive. The lack of entropies results in a lack of L^ρ–stability (integrability) for solutions u^ε of (1.3). In fact, while a maximum principle holds for the vanishing viscosity method, the thin film approximation gives much less. For n close to 2 the entropy inequality (1.2) with η_p for $p = 3-n$ gives L^ρ–stability for ρ only slightly bigger than 1. This is not even sufficient to properly define the flux $(u^\varepsilon)^n$. So the first step in our investigation will be to establish sufficiently strong L^ρ–bounds for u^ε.

The second difficulty then is to establish the entropy condition (1.2) for the limit function $u \equiv \lim_{\varepsilon\to 0} u^\varepsilon$ for *all* convex entropy–entropy flux pairs (η, q). Clearly, this property does not follow automatically from the approximation which allows only for the entropies η_p defined in (1.4). We will use an argument similar to that in [3] in which we proved that in the case of a strictly convex flux an entropy inequality for *one single* strictly convex entropy is sufficient to enforce u to be an entropy solution of the problem.

2. Main Result

The existence of solutions to the thin-film equation (1.3) and their regularity is a nontrivial issue. The usual strategy is to consider first a suitable approximation of (1.3) for which solutions exist. Some of the properties of those approximate solutions can then be passed to the limit, see [5] and [6]. We do not want to delve into these subtleties since they are not in the focus of our work. Instead we simply assume the existence of "reasonable" solutions which have all the properties we need.

Definition 2.1. Fix numbers $n \in [\frac{3}{2}, 2)$ and $\varepsilon > 0$.
A function $u \in L^1_{\text{loc}}(\mathbb{R}^+_0 \times \mathbb{R})$, $u \geq 0$ is called *admissible solution* of

$$u_t + \Big\{u^n(1 + \varepsilon u_{xxx})\Big\}_x \;=\; 0 \tag{2.1}$$

with initial data $\bar{u} \equiv u(0, \cdot)$ if the following holds

- *Initial Data*
  We have $\bar{u} \geq 0$. Then, with $p \equiv 3 - n$,

$$\int_{\mathbb{R}} \bar{u}\, dx \;=\; 1 \quad \text{and} \quad \tfrac{1}{p} \int_{\mathbb{R}} \bar{u}^p\, dx \;=\; 1\,. \tag{2.2}$$

- *Distributional Solution*
  Let $f(u) \equiv u^n$ be the flux. Then

$$\boldsymbol{u_t + f(u)_x} + \varepsilon \Big\{ \tfrac{1}{n+1} \big(u^{n+1}\big)_{xxx} - \tfrac{3n}{2} \big(u^{n-1} u_x^2\big)_x + \tfrac{n(n-1)}{2} u^{n-2} u_x^3 \Big\}_x \;=\; \boldsymbol{0} \quad \text{in } \mathcal{D}'(\mathbb{R}^+_0 \times \mathbb{R})\,. \tag{2.3}$$

  In particular, we have conservation of mass

$$\int_{\mathbb{R}} u(t)\, dx \;=\; \int_{\mathbb{R}} \bar{u}\, dx \overset{(2.2)}{=} 1 \quad \text{for a.a. } t \in \mathbb{R}^+_0\,. \tag{2.4}$$

- *Entropy Equation*
  For $p \equiv 3 - n$ define a convex entropy-entropy flux pair

$$\eta(u) \;\equiv\; \tfrac{1}{p} u^p \quad \text{and} \quad q(u) \;\equiv\; \tfrac{n}{2} u^2\,.$$

  Then, with $\mu_\varepsilon \equiv \varepsilon(p-1)\, u u_{xx}^2$,

$$\boldsymbol{\eta(u)_t + q(u)_x} + \varepsilon \Big\{ \tfrac{1}{3} \big(u^3\big)_{xxx} - \tfrac{p+5}{2} \big(u u_x^2\big)_x + \tfrac{5p+1}{6} u_x^3 \Big\}_x \;=\; \boldsymbol{-\mu}_\varepsilon \quad \text{in } \mathcal{D}'(\mathbb{R}^+_0 \times \mathbb{R})\,. \tag{2.5}$$

  In particular, we have dissipation of entropy

$$\int_{\mathbb{R}} \eta\big(u(T)\big)\, dx + \varepsilon(p-1) \int_0^T \int_{\mathbb{R}} u u_{xx}^2\, dx\, dt \;\leq\; \int_{\mathbb{R}} \eta(\bar{u})\, dx \overset{(2.2)}{=} 1 \quad \text{for a.a. } T \in \mathbb{R}^+_0\,. \tag{2.6}$$

- *Div-Curl Identity*
  If $h \in L^\infty(\mathbb{R}_0^+ \times \mathbb{R})$ is such that

$$\boldsymbol{h_x} = \boldsymbol{u}\,, \tag{2.7a}$$

$$\boldsymbol{h_t} = -\boldsymbol{f(u)} -\varepsilon\Big\{ \tfrac{1}{n+1}\big(u^{n+1}\big)_{xxx} - \tfrac{3n}{2}\big(u^{n-1}u_x^2\big)_x + \tfrac{n(n-1)}{2}u^{n-2}u_x^3 \Big\}\,, \tag{2.7b}$$

then we have

$$\begin{aligned}
&\Big(\boldsymbol{\eta(u)h}\Big)_{\boldsymbol{t}} + \Big(\boldsymbol{q(u)h}\Big)_{\boldsymbol{x}} \\
&\qquad +\varepsilon\Big\{ \tfrac{1}{3}\big(u^3h\big)_{xxx} - \tfrac{p+5}{2}\big(uu_x^2h\big)_x + \tfrac{5p+1}{6}u_x^3h - \tfrac{1}{2}\big(u^4\big)_{xx} + \tfrac{p+7}{2}u^2u_x^2 \Big\}_x \\
&= \Big(-\boldsymbol{\eta(u)f(u)} + \boldsymbol{uq(u)}\Big) - \boldsymbol{\mu_\varepsilon\, h} \\
&\qquad +\varepsilon\Big\{ \tfrac{p-1}{4p}\big(u^4\big)_{xxx} - \tfrac{p^2+8p-9}{2p}\big(u^2u_x^2\big)_x + \tfrac{4p^2+5p-9}{3p}uu_x^3 \Big\} \\
&\qquad\qquad \text{in } \mathcal{D}'(\mathbb{R}_0^+ \times \mathbb{R})\,.
\end{aligned} \tag{2.8}$$

In particular, we have the following integral identity

$$\begin{aligned}
&\int_{\mathbb{R}} \eta\big(u(T)\big)h(T)\,dx - \int_{\mathbb{R}} \eta(\bar{u})h(0)\,dx \\
&= \int_0^T \int_{\mathbb{R}} \Big(-\eta(u)f(u) + uq(u)\Big)\,dx\,dt - \varepsilon(p-1)\int_0^T\int_{\mathbb{R}} uu_{xx}^2\,h\,dx\,dt \\
&\qquad + \varepsilon\tfrac{4p^2+5p-9}{3p}\int_0^T\int_{\mathbb{R}} uu_x^3\,dx\,dt \\
&\qquad\qquad \text{for a.a. } T \in \mathbb{R}_0^+\,.
\end{aligned} \tag{2.9}$$

Remark 2.1. The identities in (2.3), (2.5) and (2.8) follow formally from straightforward manipulations of Eq. (2.1). From the entropy equation (2.6) we then obtain an $L^p(\mathbb{R})$–bound of $u(t)$ for $p = 3-n > 1$. The bound on the entropy dissipation in (2.6) will be used in Section 4.1 in numerous interpolation estimates, which prove that all terms appearing above are indeed well-defined. Note also that assumption (2.2) can always be satisfied by suitably rescaling $\bar{u}$ and x.

Remark 2.2. A function h with (2.7a)–(2.7b) exists since (2.3) implies that the second derivatives of h commute. By mass conservation we may then conclude that for a.a. $t \in \mathbb{R}_0^+$, $h(t,\cdot)$ is continuous and

$$\lim_{x\to\infty} h(t,x) - \lim_{x\to-\infty} h(t,x) = 1\,.$$

Since $f(u) + \varepsilon\big\{\cdots\big\}$ in (2.3) vanishes at infinity, Eq. (2.7b) gives

$$\lim_{x\to\infty} h(t,x) = \lim_{x\to\infty} h(0,x) \quad \text{for all } t \in [0,T]\,,$$

so we may normalize h such that $-1 \le h \le 0$.

We now state our main result.

Theorem 2.1. *Let* $\hat{u} \in L^3(\mathbb{R}_0^+ \times \mathbb{R})$ *be the unique entropy solution of the conservation law* $\hat{u}_t + f(\hat{u})_x = 0$ *with initial data* $\bar{u}$*. More precisely, assume that*

$$\eta(\hat{u})_t + q(\hat{u})_x \;\leq\; 0 \quad \text{in } D'(\mathbb{R}^+ \times \mathbb{R}) \tag{2.10}$$

for all convex entropy-entropy flux pairs (η, q)*, and*

$$\operatorname*{ess\,lim}_{t\to 0} \int_{\mathbb{R}} |\hat{u}(t) - \bar{u}|^p \, dx \;=\; 0\,. \tag{2.11}$$

Then for any $\delta > 0$ *and any compact* $K \subset \mathbb{R}_0^+ \times \mathbb{R}$*, there exists an* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *and all solutions of* $u_t + \big\{u^n(1 + \varepsilon u_{xxx})\big\}_x = 0$ *with initial data* $\bar{u}$ *in the sense of Definition 2.1, we have*

$$\iint_K |u - \hat{u}|^3 \, dx\, dt \;\leq\; \delta\,.$$

Remark 2.3. A similar result can be proved by assuming the entropy equation (2.5) for an entropy η_p as in (1.4) with $p \leq 1 + \frac{n}{3}$. This also allows to consider the case with $n \in \big(1, \frac{3}{2}\big)$, but we will not pursue this idea here.

3. Preliminaries

We need some preparations.

3.1. **Young measures.** In the proof of Theorem 2.1 we will use Young measures to describe the possible limits of nonlinear compositions of sequences $\{u_\nu\}_\nu$ which are bounded in some L^ρ–space with $\rho < \infty$. We therefore need generalized Young measures, introduced by DiPerna & Majda [4], which can handle both oscillations and concentrations.

Definition 3.1. Let $C(\mathbb{R}_0^+)$ be the usual space of continuous functions on $\mathbb{R}_0^+$ equipped with the sup-norm. For $\rho \in [0, \infty)$ we define

$$C_\rho(\mathbb{R}_0^+) \;\equiv\; \Big\{ g \in C(\mathbb{R}_0^+) \;\Big|\; g_* \equiv \lim_{s\to\infty} \tfrac{g(s)}{(1+|s|)^\rho} \text{ exists} \Big\}\,.$$

Let $BC(\mathbb{R}_0^+)$ be the space of bounded continuous functions.

Note that $C_0(\mathbb{R}_0^+)$ is a completely regular subalgebra of $BC(\mathbb{R}_0^+)$, i.e., it is closed with respect to the sup-norm, contains constants, and separates points. Associated with any such subalgebra is a compactification of $\mathbb{R}_0^+$. For $C_0(\mathbb{R}_0^+)$ we obtain the one-point compactification of $\mathbb{R}_0^+$ denoted by $\gamma\mathbb{R}_0^+$. This $\gamma\mathbb{R}_0^+$ is a compact completely regular Hausdorff space, and the subalgebra $C_0(\mathbb{R}_0^+)$ is isomorphic with $C(\gamma\mathbb{R}_0^+)$. The continuous linear functionals of $C_0(\mathbb{R}_0^+)$ can thus be identified with measures in $\mathcal{M}(\gamma\mathbb{R}_0^+)$. In fact, they admit a decomposition into a measure in $\mathcal{M}(\mathbb{R}_0^+)$ and a Dirac mass at ∞. We refer to [4, 8].

Proposition 3.1. *For $\Omega \subset Q_T$ a bounded Borel set, let $\{u_\nu\}_\nu$ be a sequence of nonnegative functions which are uniformly bounded in $L^\rho(\Omega)$ for some $\rho \in [1,\infty)$. Then there exist a pair $(\mu, \sigma_{con}) \in L^\infty_{\mathrm{w}}\big(\Omega; \mathrm{Prob}(\mathbb{R}^+_0)\big) \times \mathcal{M}_+(\bar\Omega)$ (i.e., with μ a weakly measurable mapping from Ω into the space of probability measures on $\mathbb{R}^+_0$) and a subsequence (denoted again by $\{u_\nu\}_\nu$), such that for all $g \in C_\rho(\mathbb{R}^+_0)$*

$$g(u_\nu) \stackrel{*}{\rightharpoonup} \langle g\rangle + g_* \sigma_{con} \quad \text{in } \mathcal{M}(\bar\Omega)\,,$$

with function $\langle g\rangle \in L^1(\Omega)$ defined by

$$\langle g\rangle_z \;\equiv\; \int_{\mathbb{R}^+_0} g(s)\,\mu_z(ds) \quad \text{for all } z \in \Omega\,.$$

The measure σ_{con} captures the concentrations in the limit.

Proof. We adapt the proof of Theorem 3.2.13 in [8] to show that for any $\{u_\nu\}_\nu$ as above, there exist a pair $(\sigma, \hat\mu) \in \mathcal{M}_+(\bar\Omega) \times L^\infty_{\mathrm{w}}\big(\bar\Omega, \sigma; \mathrm{Prob}(\gamma\mathbb{R}^+_0)\big)$ (i.e., with $\hat\mu$ a weakly σ-measurable mapping from $\bar\Omega$ into the space of probability measures on $\gamma\mathbb{R}^+_0$) and a subsequence (denoted again by $\{u_\nu\}_\nu$), such that for all $g \in C_\rho(\mathbb{R}^+_0)$

$$g(u_\nu) \stackrel{*}{\rightharpoonup} \hat g\sigma \quad \text{in } \mathcal{M}(\bar\Omega)\,,$$

with function $\hat g \in L^\infty(\bar\Omega, \sigma)$ given by

$$\hat g(z) \;\equiv\; \int_{\gamma\mathbb{R}^+_0} g_0(s)\,\hat\mu_z(ds) \quad \text{for } \sigma\text{-a.a. } z \in \bar\Omega\,.$$

Here, $g_0(s) \equiv \frac{g(s)}{1+|s|^\rho}$ if $s \in \mathbb{R}^+_0$ and $g_0(\infty) \equiv g_*$. In the second step we decompose the limit $\hat g\sigma$ into an absolutely continuous part, and a part which captures possible concentrations in the sequence $\{g(u_\nu)\}_\nu$. We first note that the measure

$$\hat\sigma(dz) \;\equiv\; \bigg(\int_{\mathbb{R}^+_0} \hat\mu_z(ds)\bigg)\sigma(dz)$$

is absolutely continuous with respect to the Lebesgue measure, and that indeed $\int_{\mathbb{R}^+_0} \hat\mu_z(ds) > 0$ for a.a. $z \in \Omega$, see Theorem 3.2.15 in [8]. Thus we can define a new family $\{\mu_z\}_{z\in\Omega}$ of measures by putting

$$\mu_z(ds) \;\equiv\; \bigg(\int_{\mathbb{R}^+_0} \frac{\hat\mu_z(ds)}{1+|s|^\rho}\bigg)^{-1} \frac{\hat\mu_z|_{\mathbb{R}^+_0}(ds)}{1+|s|^\rho} \quad \text{for a.a. } z \in \Omega\,.$$

This is the μ of the proposition, and then $\sigma_{con} \equiv \hat\mu\big(\{\infty\}\big)\,\sigma$. □

3.2. **Commutator.** We assume a strictly convex flux f.

Definition 3.2. Let $\mu \in \mathrm{Prob}(\mathbb{R}^+_0)$, and

$$\langle g\rangle \;\equiv\; \int g(s)\,d\mu(s) \quad \text{for } g \in L^1(\mathbb{R}^+_0, \mu)\,.$$

For any convex entropy η with $\eta(0) = 0$ and $\eta(\infty) = \infty$, and corresponding entropy flux q, we define the bilinear form

$$B(\eta, f) \;\equiv\; \big\langle uq(u) - \eta(u)f(u)\big\rangle - \big\langle u\big\rangle\big\langle q(u)\big\rangle + \big\langle \eta(u)\big\rangle\big\langle f(u)\big\rangle\,.$$

Note that $B(\eta, f)$ has the structure of a commutator.

Then we have the following result

Lemma 3.2. *Let η be a strictly convex entropy as in Definition 3.2, and define the bilinear form $B(\eta, f)$. Then*

(i) $B(\eta, f) \;\geq\; 0$;
(ii) $B(\eta, f) \;=\; 0$ *if and only if* $\mu = \delta_c$ *for some* $c \in \mathbb{R}_0^+$, *i.e., the probability measure* μ *is concentrated in one point.*

Proof. Since η is strictly convex, the second distributional derivative is a positive measure on $\mathbb{R}_0^+$. Thus we have the following representation

$$\eta(u) \;=\; \int_{\mathbb{R}_0^+} (u-s)_+\,\eta''(ds) \quad \text{for all } u \in \mathbb{R}_0^+\,.$$

Since B is linear in η, we can write

$$B(\eta, f) \;=\; \int_{\mathbb{R}_0^+} B(\eta_s, f)\,\eta''(ds) \quad \text{with } \eta_s(u) \equiv (u-s)_+\,. \tag{3.1}$$

The corresponding entropy flux q_s is given by

$$q_s(u) \;\equiv\; \big(f(u) - f(s)\big)H(u-s) \quad \text{with } H \text{ the Heaviside function}\,.$$

Then some straightforward computation gives

$$\begin{aligned} B(\eta_s, f) \;\;=\;\; &\int_{\{u<s\}} f(u) - f(s) - f'(s)(u-s)\,d\mu \cdot \int_{\{u>s\}} u - s\,d\mu \\ &- \int_{\{u>s\}} f(u) - f(s) - f'(s)(u-s)\,d\mu \cdot \int_{\{u<s\}} u - s\,d\mu\,. \end{aligned}$$

Convexity of f implies $f(u) - f(s) - f'(s)(u-s) \geq 0$ for all $u, s \in \mathbb{R}_0^+$. Hence $B(\eta_s, f) \geq 0$ for all s, and so (i) follows with (3.1).

The strict convexity of f implies that $B(\eta_s, f) > 0$ if and only if

$$\mu\big([0,s)\big) \cdot \mu\big((s,\infty)\big) > 0\,.$$

Thus $B(\eta, f) = 0$ if and only if $B(\eta_s, f) = 0$ for η''-a.a. $s \in \mathbb{R}_0^+$, which can happen if and only if $\mu\big([0,s)\big) = 0$ or $\mu\big((s,\infty)\big) = 0$. But

$$\Big(\mu\big([0,s_0)\big) \;=\; 0\Big) \quad\Longrightarrow\quad \Big(\mu\big([0,s)\big) \;=\; 0 \quad \text{for all } s \leq s_0\Big)\,,$$

with a similar statement for $\mu\big((s,\infty)\big)$. We conclude that $B(\eta, f) = 0$ if and only if there exists a $c \in \mathbb{R}_0^+$ with $\mu\big(\{c\}\big) = 1$. This is (ii). □

4. Proof of Theorem 2.1: Convergence

We use an indirect argument. Assume indeed the theorem is false. Then there exist $\delta > 0$, a compact $K \subset \mathbb{R}_0^+ \times \mathbb{R}$, a sequence $\varepsilon_\nu \to 0$ and a sequence $\{u_\nu\}_\nu$ of admissible solutions of $u_t + \big\{u^n(1+\varepsilon u_{xxx})\big\}_x = 0$ in the sense of Definition 2.1 such that

$$\iint_K |u_\nu - \hat{u}|^3 \, dx \, dt \;\geq\; \delta \, .$$

We will show that a subsequence of $\{u_\nu\}_\nu$ converges in $L^3_{\mathrm{loc}}(\mathbb{R}_0^+ \times \mathbb{R})$ to the unique entropy solution $\hat{u}$, which gives a contradiction. We split the proof into two parts. In this section we establish strong convergence for a subsequence of $\{u_\nu\}_\nu$; in the next section we prove that conditions (2.10) and (2.11) are satisfied for the limit.

We proceed in several steps.

- In Section 4.1 we establish several interpolation estimates.
- In Section 4.2 we provide uniform higher integrability for $\{u_\nu\}_\nu$.
- In Section 4.3 we prove that a subsequence of $\{u_\nu\}_\nu$ converges to a measure-valued solution of $\hat{u}_t + f(\hat{u})_x = 0$.
- In Section 4.4 we prove that convergence is actually strong. Then we can pass to the limits in the nonlinear terms.

For convenience, we will use the notation $A \lesssim B$ to indicate the existence of a universal constant $C > 0$ such that $A \leq CB$.

4.1. **Interpolation.** We start with some preparatory estimates.

Lemma 4.1. *For a function $v \geq 0$ and $r \in [1,7]$, it holds*

$$\int_{\mathbb{R}} v^r \, dx \;\lesssim\; \left(\int_{\mathbb{R}} v \, dx\right)^{\frac{r+1}{2}} \left(\int_{\mathbb{R}} v v_{xx}^2 \, dx\right)^{\frac{r-1}{6}} , \tag{4.1}$$

$$\int_{\mathbb{R}} v^{\frac{r-3}{4}} |v_x^3| \, dx \;\lesssim\; \left(\int_{\mathbb{R}} v \, dx\right)^{\frac{r+1}{8}} \left(\int_{\mathbb{R}} v v_{xx}^2 \, dx\right)^{\frac{r+17}{24}} , \tag{4.2}$$

$$\int_{\mathbb{R}} v^{\frac{r-1}{2}} v_x^2 \, dx \;\lesssim\; \left(\int_{\mathbb{R}} v \, dx\right)^{\frac{r+1}{4}} \left(\int_{\mathbb{R}} v v_{xx}^2 \, dx\right)^{\frac{r+5}{12}} . \tag{4.3}$$

Proof. We first derive two preliminary estimates. We have

$$\int_{\mathbb{R}} v^{\frac{r-3}{4}} |v_x^3| \, dx \;=\; \tfrac{4}{r+1} \int_{\mathbb{R}} \big(v^{\frac{r+1}{4}}\big)_x |v_x| v_x \, dx \;=\; -\tfrac{8}{r+1} \int_{\mathbb{R}} v^{\frac{r+1}{4}} |v_x| v_{xx} \, dx \, ,$$

after integration by parts. Then Hölder inequality gives

$$\int_{\mathbb{R}} v^{\frac{r-3}{4}} |v_x^3| \, dx \;\lesssim\; \left(\int_{\mathbb{R}} v^r \, dx\right)^{1/6} \left(\int_{\mathbb{R}} v^{\frac{r-3}{4}} |v_x^3| \, dx\right)^{1/3} \left(\int_{\mathbb{R}} v v_{xx}^2 \, dx\right)^{1/2} ,$$

hence

$$\int_{\mathbb{R}} v^{\frac{r-3}{4}} |v_x^3| \, dx \;\lesssim\; \left(\int_{\mathbb{R}} v^r \, dx\right)^{1/4} \left(\int_{\mathbb{R}} v v_{xx}^2 \, dx\right)^{3/4} . \tag{4.4}$$

Using again Hölder inequality we obtain

$$\begin{aligned} \int_{\mathbb{R}} v^{\frac{r-1}{2}} v_x^2 \, dx \;&\le\; \left(\int_{\mathbb{R}} v^r \, dx\right)^{1/3} \left(\int_{\mathbb{R}} v^{\frac{r-3}{4}} |v_x^3| \, dx\right)^{2/3} \\ &\lesssim\; \left(\int_{\mathbb{R}} v^r \, dx\right)^{1/2} \left(\int_{\mathbb{R}} v v_{xx}^2 \, dx\right)^{1/2} . \end{aligned} \tag{4.5}$$

Now we prove (4.1). We start with the following pointwise estimate

$$\big(\sup v\big)^2 \;\le\; 2 \int_{\mathbb{R}} v |v_x| \, dx \;\lesssim\; \left(\int_{\mathbb{R}} v \, dx\right)^{2/3} \left(\int_{\mathbb{R}} v |v_x^3| \, dx\right)^{1/3} ,$$

which implies

$$\int_{\mathbb{R}} v^7 \, dx \;\le\; \big(\sup v\big)^6 \int_{\mathbb{R}} v \, dx \;\lesssim\; \left(\int_{\mathbb{R}} v \, dx\right)^3 \int_{\mathbb{R}} v |v_x^3| \, dx . \tag{4.6}$$

Using (4.4) in (4.6) (with $r = 7$) and arranging terms give

$$\int_{\mathbb{R}} v^7 \, dx \;\lesssim\; \left(\int_{\mathbb{R}} v \, dx\right)^4 \int_{\mathbb{R}} v v_{xx}^2 \, dx . \tag{4.7}$$

Now we use Hölder inequality once again to estimate for $r \in [1, 7]$

$$\int_{\mathbb{R}} v^r \, dx \;\le\; \left(\int_{\mathbb{R}} v \, dx\right)^{\frac{7-r}{6}} \left(\int_{\mathbb{R}} v^7 \, dx\right)^{\frac{r-1}{6}} . \tag{4.8}$$

Then (4.1) follows from (4.7) and (4.8). Finally, (4.1) implies the estimates (4.2) and (4.3) because of (4.4) and (4.5). □

Lemma 4.2. *Let u be an admissible solution of* (2.1)*. Then*

$$\varepsilon \iint_{\mathbb{R}_0^+ \times \mathbb{R}} u |u_x^3| \, dx \, dt \;\lesssim\; 1 .$$

Proof. For $r = 7$ the estimate (4.2) in Lemma 4.1 simplifies to

$$\int_{\mathbb{R}} v |v_x^3| \, dx \;\lesssim\; \int_{\mathbb{R}} v \, dx \int_{\mathbb{R}} v v_{xx}^2 \, dx .$$

We put $v \equiv u(t, \cdot)$ and integrate in time, using that $\int_{\mathbb{R}} u(t) \, dx = 1$ for a.a. $t \in \mathbb{R}_0^+$ by mass conservation (2.4). Then the claim follows from estimate (2.6). □

Lemma 4.3. *Let $R \equiv [t_1, t_2) \times \mathbb{R}$, and let u be an admissible solution of* (2.1)*. Then we have, as $\varepsilon |t_2 - t_1|$ tends to zero, for all $r \in [1, 7]$*

$$\varepsilon \iint_R u^r \, dx \, dt + \varepsilon \iint_R u^{\frac{r-3}{4}} |u_x^3| \, dx \, dt + \varepsilon \iint_R u^{\frac{r-1}{2}} u_x^2 \, dx \, dt \;=\; o(1) . \tag{4.9}$$

Proof. For $r \in [1,7)$ our claim follows from Lemma 4.1, in combination with mass conservation (2.4) and entropy estimate (2.6). We have, e.g.,

$$\begin{aligned} \varepsilon \iint_R u^r \, dx \, dt &\lesssim \varepsilon \int_{t_1}^{t_2} \left(\int_{\mathbb{R}} u \, dx \right)^{\frac{r+1}{2}} \left(\int_{\mathbb{R}} u u_{xx}^2 \, dx \right)^{\frac{r-1}{6}} dt \\ &\lesssim \big(\varepsilon |t_2 - t_1| \big)^{\frac{7-r}{6}} \left(\varepsilon \iint_{\mathbb{R}_0^+ \times \mathbb{R}} u u_{xx}^2 \, dx \, dt \right)^{\frac{r-1}{6}} . \end{aligned}$$

We used Hölder inequality for the integration in time. The estimates for the other quantities $\varepsilon \iint_R u^{\frac{r-3}{4}} |u_x^3| \, dx \, dt$ and $\varepsilon \iint_R u^{\frac{r-1}{2}} u_x^2 \, dx \, dt$ can be handled in a similar way. For the extreme case $r = 7$ we modify (4.7) according to

$$\begin{aligned} \int_{\mathbb{R}} v^7 \, dx &\leq \big(\sup v \big)^{7-p} \int_{\mathbb{R}} v^p \, dx \\ &\lesssim \left(\int_{\mathbb{R}} v \, dx \right)^{\frac{7-p}{3}} \left(\int_{\mathbb{R}} v |v_x^3| \, dx \right)^{\frac{7-p}{6}} \int_{\mathbb{R}} v^p \, dx . \end{aligned} \tag{4.10}$$

Using (4.4) in (4.10) and arranging terms then yield

$$\int_{\mathbb{R}} v^7 \, dx \lesssim \left(\int_{\mathbb{R}} v \, dx \right)^{\frac{56-8p}{17+p}} \left(\int_{\mathbb{R}} v^p \, dx \right)^{\frac{24}{17+p}} \left(\int_{\mathbb{R}} v v_{xx}^2 \, dx \right)^{\frac{21-3p}{17+p}} .$$

Note that $\frac{21-3p}{17+p} < 1$ since $p = 3 - n > 1$. We then obtain

$$\begin{aligned} &\varepsilon \iint_R u^7 \, dx \, dt \\ &\quad \lesssim \varepsilon \int_{t_1}^{t_2} \left(\int_{\mathbb{R}} u \, dx \right)^{\frac{56-8p}{17+p}} \left(\int_{\mathbb{R}} u^p \, dx \right)^{\frac{24}{17+p}} \left(\int_{\mathbb{R}} u u_{xx}^2 \, dx \right)^{\frac{21-3p}{17+p}} dt \\ &\quad \lesssim \big(\varepsilon |t_2 - t_1| \big)^{\frac{4p-4}{17+p}} \left(\varepsilon \iint_{\mathbb{R}_0^+ \times \mathbb{R}} u u_{xx}^2 \, dx \, dt \right)^{\frac{21-3p}{17+p}} , \end{aligned}$$

using mass conservation (2.4) and entropy inequality (2.6). The estimates for $\varepsilon \iint_R u |u_x^3| \, dx \, dt$ and $\varepsilon \iint_R u^3 u_x^2 \, dx \, dt$ follow in an analogous way. □

4.2. **Integrability.** Here we prove uniform higher integrability.

Proposition 4.4. *Let u be an admissible solution of* (2.1). *Then there exists a constant C which does not depend on ε, such that*

$$\iint_{\mathbb{R}_0^+ \times \mathbb{R}} u^3 \, dx \, dt \leq C . \tag{4.11}$$

Proof. We derive this proposition from the div-curl identity (2.9). By Remark 2.2, we can assume that h is normalized such that $-1 \leq h \leq 0$. Note that

$$-\eta(u) f(u) + u q(u) = c_n u^3 \quad \text{with } c_n \equiv -\tfrac{1}{p} + \tfrac{n}{2} > 0 \tag{4.12}$$

which is the quantity we want to control. Then (4.11) follows from the entropy inequality (2.6) and from Lemma 4.2. □

4.3. **Compactness.** From Proposition 4.4 we know that the sequence $\{u_\nu\}_\nu$ is uniformly bounded in $L^3(\mathbb{R}_0^+ \times \mathbb{R})$. Here we will prove that a subsequence converges to a measure-valued solution of $\hat{u}_t + f(\hat{u})_x = 0$, and prepare the proof of strong convergence.

Lemma 4.5. *For each ν let $h_\nu \in L^\infty(\mathbb{R}_0^+ \times \mathbb{R})$ be defined by* (2.7a)–(2.7b) *with u replaced by u_ν, normalized such that $-1 \le h_\nu \le 0$. Then the sequence $\{h_\nu\}_\nu$ is equicontinuous on $\mathbb{R}_0^+ \times \mathbb{R}$.*

Proof. For ease of notation, we do not write the subscript ν in the following. From entropy inequality (2.6) we have integrability of $u^p(t,\cdot)$ with $p = 3-n > 1$ for a.a. $t \in \mathbb{R}_0^+$, uniformly in ε. Therefore

$$\big|h(t,y_2) - h(t,y_1)\big| \;\le\; \int_{y_1}^{y_2} |h_x(t)|\,dx \;\lesssim\; |y_2 - y_1|^{1/p'} \tag{4.13}$$

for any $y_1, y_2 \in \mathbb{R}$, which gives continuity in space. To prove continuity in time we fix a $\varphi \in \mathcal{D}(\mathbb{R})$ with $\varphi \ge 0$ and $\int_{\mathbb{R}} \varphi\,dx = 1$. Then we put

$$\varphi_\delta(x) \;\equiv\; \tfrac{1}{\delta}\varphi(\tfrac{x}{\delta}) \quad \text{for } x \in \mathbb{R} \text{ and } \delta > 0\,.$$

The constant δ will be optimized later. We use (4.13) to estimate

$$\begin{aligned} &\left|\left(\int_{\mathbb{R}} \varphi_\delta(x-y) h(t,x)\,dx\right) - h(t,y)\right| \\ &\qquad \lesssim\; \int_{\mathbb{R}} \varphi_\delta(x-y) |x-y|^{1/p'}\,dx \;\lesssim\; \delta^{1/p'} \end{aligned}$$

for $(t,y) \in \mathbb{R}_0^+ \times \mathbb{R}$. Then we find

$$\begin{aligned} &\big|h(t_2,y) - h(t_1,y)\big| \\ &\quad \lesssim\; \delta^{1/p'} + \left|\int_{\mathbb{R}} \varphi_\delta(x-y)\big[h(t_2,x) - h(t_1,x)\big]\,dx\right| \\ &\quad =\; \delta^{1/p'} + \left|\int_{t_1}^{t_2}\int_{\mathbb{R}} \varphi_\delta(x-y) h_t(t,x)\,dx\,dt\right| \end{aligned}$$

for $t_1, t_2 \in \mathbb{R}_0^+$ and $y \in \mathbb{R}$. Although h_t is defined only in distributional sense by (2.7b), all terms can be controlled. We have, e.g.,

$$\begin{aligned} &\left|\int_{t_1}^{t_2}\int_{\mathbb{R}} \varphi_\delta(x-y)\, u^n(t,x)\,dx\,dt\right| \\ &\quad \lesssim\; \left(\int_{t_1}^{t_2}\int_{\mathbb{R}} \big|\varphi_\delta(x-y)\big|^{\frac{3}{3-n}}\,dx\,dt\right)^{\frac{3-n}{3}} \;\lesssim\; |t_2 - t_1|^{\frac{3-n}{3}}\,\delta^{-1+\frac{3-n}{3}}\,, \end{aligned}$$

uniformly in ε. We used Hölder inequality and Proposition 4.4. Similarly, we can estimate the terms in curly brackets in (2.7b) uniformly in ε, using Lemma 4.3. Note that for each x–derivative an extra factor δ^{-1} appears after integration by parts. However, choosing first δ, then $|t_2 - t_1|$ small enough, we conclude that h is continuous in time, with a modulus of continuity which is uniform in ε. $\square$

Consider now a sequence $\{K_m\}_m$ of compact sets $K_m \uparrow (\mathbb{R}_0^+ \times \mathbb{R})$. On each K_m the Arzelà-Ascoli Theorem asserts the existence of a subsequence of $\{h_\nu\}_\nu$ converging uniformly to a continuous function. Then we obtain, by a diagonal argument, a function $\hat{h} \in C(\mathbb{R}_0^+ \times \mathbb{R})$ with

$$h_\nu \longrightarrow \hat{h} \quad \text{uniformly on compact sets}\,. \tag{4.14}$$

Moreover, if $-1 \le h_\nu \le 0$ for all ν, then also $-1 \le \hat{h} \le 0$.

Now we use Proposition 3.1 with $\Omega \equiv (0,T) \times \mathcal{O}$, with $T > 0$, $\mathcal{O} \subset \mathbb{R}$ open, and $\rho \equiv 3$. Then we have the following convergence

$$-\eta(u_\nu)f(u_\nu) + u_\nu q(u_\nu) \overset{*}{\rightharpoonup} \big\langle uq(u) - \eta(u)f(u)\big\rangle + c_n\sigma_{con} \tag{4.15}$$

in $\mathcal{M}(\bar{\Omega})$, see (4.12). On the other hand, we have

$$\left.\begin{array}{rcl} u_\nu & \rightharpoonup & \langle u\rangle \\ \eta(u_\nu) & \rightharpoonup & \big\langle \eta(u)\big\rangle \\ f(u_\nu) & \rightharpoonup & \big\langle f(u)\big\rangle \\ q(u_\nu) & \rightharpoonup & \big\langle q(u)\big\rangle \end{array}\right\} \quad \text{weakly in } L^1(\Omega) \tag{4.16}$$

since these functions have growth strictly less than cubic at infinity. It is then straightforward to check that the terms in curly brackets in both (2.3) and (2.5) vanish in distributional sense because of Lemma 4.3. Note that the most singular term to estimate is $u^{n-2}u_x^3$, where $n-2 < 0$. This can be controlled by (4.9) only if $n \ge \frac{3}{2}$. Extracting another subsequence if necessary, we may also assume that

$$\mu_{\varepsilon_\nu} \rightharpoonup \hat{\mu} \quad \text{in } \mathcal{M}(\bar{\Omega})\,, \tag{4.17}$$

for some nonnegative measure $\hat{\mu}$. Then it is easily seen that

$$\left.\begin{array}{rcl} \langle u\rangle_t + \big\langle f(u)\big\rangle_x & = & 0 \\ \big\langle \eta(u)\big\rangle_t + \big\langle q(u)\big\rangle_x & = & -\hat{\mu} \end{array}\right\} \quad \text{in } \mathcal{D}'(\Omega)\,. \tag{4.18}$$

Moreover, (4.14) implies that $\hat{h}$ satisfies

$$\hat{h}_t = -\big\langle f(u)\big\rangle \quad \text{and} \quad \hat{h}_x = \langle u\rangle\,. \tag{4.19}$$

Choosing now a sequence $\{\Omega_m\}_m$ with $\Omega_m \uparrow (\mathbb{R}_0^+ \times \mathbb{R})$ proves the claim.

4.4. **Strong convergence.** Let $\Omega \equiv (0,T)\times\mathbb{R}$ with $T>0$. To prove convergence of $\{u_\nu\}_\nu$ in $L^3(\Omega)$ we use

$$u \;=\; u_\nu\,, \quad h \;=\; h_\nu \quad \text{and} \quad \varepsilon \;=\; \varepsilon_\nu$$

in the div-curl identity (2.8) and pass to the limit $\nu\to\infty$. We conclude from (4.14)–(4.17) that the following new integral identity holds

$$\begin{aligned}\iint_\Omega \big\langle uq(u)-\eta(u)f(u)\big\rangle\,\xi\,dx\,dt + c_n\iint_\Omega \xi\,\sigma_{con}(dx,dt)\\ = \; -\iint_\Omega \hat\mu\hat h\,\xi\,dx\,dt - \iint_\Omega \Big\{\big\langle\eta(u)\big\rangle\,\xi_t + \big\langle q(u)\big\rangle\,\xi_x\Big\}\,\hat h\,dx\,dt\,,\end{aligned}$$

for any $\xi\in\mathcal{D}(\Omega)$. In fact, the terms in curly brackets in (2.8) disappear because of Lemma 4.3. Integration by parts and (4.18)–(4.19) then give

$$\begin{aligned}\iint_\Omega \Big\{\big\langle uq(u)-\eta(u)f(u)\big\rangle - \langle u\rangle\big\langle q(u)\big\rangle + \big\langle\eta(u)\big\rangle\big\langle f(u)\big\rangle\Big\}\,\xi\,dx\,dt\\ + c_n\iint_\Omega \xi\,\sigma_{con}(dx,dt) \;=\; 0\,.\end{aligned}$$

Let ξ be nonnegative. Then Lemma 3.2 (i) implies that the first integral is nonnegative. Hence both terms on the left-hand side must be zero. From Lemma 3.2 (ii) we conclude that

(1) there is no concentration, i.e., $\sigma_{con}=0$;
(2) for a.a. $(t,x)\in\Omega$, the Young measure is concentrated at one point, and so convergence of $\{u_\nu\}_\nu$ is strong in $L^3(\Omega)$.

We define the measurable function $\hat u\equiv\langle u\rangle$. Then

$$\big\langle\eta(u)\big\rangle \;=\; \eta(\hat u), \quad \big\langle f(u)\big\rangle \;=\; f(\hat u) \quad\text{and}\quad \big\langle q(u)\big\rangle \;=\; q(\hat u)\,.$$

Since ξ and Ω were arbitrary, this gives strong convergence of a subsequence of $\{u_\nu\}_\nu$ in $L^3_{\mathrm{loc}}(\mathbb{R}^+_0\times\mathbb{R})$ to a weak solution of $\hat u_t + f(\hat u)_x = 0$.

5. Proof of Theorem 2.1: Uniqueness

The proof consist of two steps.

- In Section 5.1 we prove that $\hat u$ satisfies entropy condition (2.10).
- In Section 5.2 we show that the initial data is assumed in $L^p(\mathbb{R})$.

Then the uniqueness of $\hat u$ follows from Kružkov theory, see [7].

5.1. **Entropy condition.** The proof is a variant of the argument given in [3]. We repeat the main ideas for the reader's convenience and briefly comment on the necessary adaptations. We use the theory of *viscosity solutions* of Hamilton-Jacobi equations, see [2].

Definition 5.1. Let $\Omega\subset\mathbb{R}^2$ be open, and f a flux. A function $h\in C(\Omega)$ is called a viscosity solution of $h_t+f(h)_x=0$ if for any point $(t,x)\in\Omega$ the following is true: For any smooth function ζ,

(i) if $h-\zeta$ has a local maximum at (t,x), then

$$\zeta_t(t,x) + f\big(\zeta_x(t,x)\big) \;\leq\; 0\,;$$

(ii) if $h-\zeta$ has a local minimum at (t,x), then

$$\zeta_t(t,x) + f\big(\zeta_x(t,x)\big) \;\geq\; 0\,.$$

It is well-known that h is a viscosity solution of $h_t + f(h_x) = 0$ if and only if $u \equiv h_x$ is an entropy solution of the conservation law $u_t + f(u)_x = 0$. This follows from the fact that both solutions are obtained by the vanishing viscosity method.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open subset, and assume that $u \in L^\rho(\Omega)$, with exponent $1 \leq \rho < \infty$, satisfies*

$$\left.\begin{aligned} u_t + f(u)_x &= 0 \\ \eta(u)_t + q(u)_x &\leq 0 \end{aligned}\right\} \quad \text{in } \mathcal{D}'(\Omega)\,, \tag{5.1}$$

for a strictly convex flux f and one strictly convex entropy-entropy flux pair (η,q). Assume that $uq(u) - \eta(u)f(u) \geq C(1+|u|^\rho)$ for large u, and that f,η and q have growth strictly less than u^ρ at infinity. If now $h \in C(\Omega)$ is defined by

$$h_t \;=\; -f(u) \quad \text{and} \quad h_x \;=\; u\,, \tag{5.2}$$

then h is a viscosity solution of $h_t + f(h_x) = 0$.

Remark 5.1. Condition (2.10) for $\hat{u}$ is a corollary of Proposition 5.1.

Proof. Let ζ be a smooth function, and $(t,x) \in \Omega$. Statement (i) of Definition 5.1 above then follows from Jensen inequality and a stability result for viscosity (sub)solutions. We refer to [3]. So we only consider the case when $h-\zeta$ has a local minimum at (t,x), and we will show that statement (ii) then holds even with equality. Without loss of generality we may assume that $(t,x) = (0,0)$ and $\big(h-\zeta\big)(0,0) = 0$, and that the minimum is strict. For any $\alpha > 0$ we consider

$$\Omega_\alpha \;\equiv\; \text{connected component of } \big\{h - \zeta < \alpha\big\} \text{ containing } (0,0)\,.$$

Since h is continuous and $(0,0)$ is a strict minimum, Ω_α is an open set and $\operatorname{diam}(\Omega_\alpha) \to 0$ as $\alpha \to 0$. For any $g \in C_\rho(\mathbb{R}_0^+)$ we now define

$$\big\langle g(u)\big\rangle_\alpha \;\equiv\; \frac{1}{|\Omega_\alpha|} \iint_{\Omega_\alpha} g\big(u(t,x)\big)\,dt\,dx\,.$$

Note that $\big\langle g(u)\big\rangle_\alpha$ is well-defined for positive α since $u \in L^\rho(\Omega)$. As in [3] we can then prove that by choice of ζ and because of (5.2)

$$-\big\langle f(u)\big\rangle_\alpha \;=\; \big\langle \zeta_t\big\rangle_\alpha \quad \text{and} \quad \big\langle u\big\rangle_\alpha \;=\; \big\langle \zeta_x\big\rangle_\alpha\,, \tag{5.3}$$

and that (5.1) in combination with Lemma 3.2 (i) gives

$$\lim_{\alpha\to 0} \Big| \big\langle uq(u) - \eta(u)f(u)\big\rangle_\alpha - \big\langle u\big\rangle_\alpha \big\langle q(u)\big\rangle_\alpha + \big\langle \eta(u)\big\rangle_\alpha \big\langle f(u)\big\rangle_\alpha \Big| \;=\; 0\,. \tag{5.4}$$

We now define a family of measures $\hat{\mu}_\alpha \in \mathrm{Prob}(\gamma\mathbb{R}_0^+)$ by

$$\int_{\gamma\mathbb{R}_0^+} g_0(s)\,\hat{\mu}_\alpha(ds) \;\equiv\; \frac{\big\langle g(u)\big\rangle_\alpha}{\big\langle 1+|u|^\rho\big\rangle_\alpha} \tag{5.5}$$

for $g_0 \in C(\gamma\mathbb{R}_0^+)$ and $g(s) \equiv (1+|s|^\rho)g_0(s)$ if $s \in \mathbb{R}_0^+$. Testing with the constant function $g \equiv 1$ we conclude that $\big\langle 1+|u|^\rho\big\rangle_\alpha$ can blow up as $\alpha \to 0$ if and only if $\{\hat{\mu}_\alpha\}_\alpha$ concentrates all mass at ∞. But this is impossible: Since ζ is smooth

$$\big\langle \zeta_t\big\rangle_\alpha \;\longrightarrow\; \zeta_t(0,0) \quad\text{and}\quad \big\langle \zeta_x\big\rangle_\alpha \;\longrightarrow\; \zeta_x(0,0) \tag{5.6}$$

as $\alpha \to 0$, and so (5.3) implies that $\big\langle f(u)\big\rangle_\alpha$ and $\big\langle u\big\rangle_\alpha$ stay bounded. By assumption, η and q have growth strictly less than u^ρ, and then (5.5) yields

$$-\big\langle u\big\rangle_\alpha\big\langle q(u)\big\rangle_\alpha + \big\langle \eta(u)\big\rangle_\alpha\big\langle f(u)\big\rangle_\alpha \;=\; o(1)\,\big\langle 1+|u|^\rho\big\rangle_\alpha$$

as $\alpha \to 0$. Moreover, we have

$$\big\langle uq(u) - \eta(u)f(u)\big\rangle_\alpha \;\geq\; C\big\langle 1+|u|^\rho\big\rangle_\alpha\,.$$

Therefore, blow-up of $\big\langle 1+|u|^\rho\big\rangle_\alpha$ would contradict (5.4).

Consider any subsequence of $\{\hat{\mu}_\alpha\}_\alpha$ converging weak* in $\mathcal{M}(\gamma\mathbb{R}_0^+)$ to a limit measure $\hat{\mu} \in \mathrm{Prob}(\gamma\mathbb{R}_0^+)$. Such a subsequence always exists by Banach-Alaoglu Theorem. Let $\sigma \equiv \lim_{\alpha\to 0}\big\langle 1+|u|^\rho\big\rangle_\alpha$. Then

$$\sigma \;=\; \left(\int_{\gamma\mathbb{R}_0^+} \frac{\hat{\mu}(ds)}{1+|s|^\rho}\right)^{-1} \;=\; \left(\int_{\mathbb{R}_0^+} \frac{\hat{\mu}(ds)}{1+|s|^\rho}\right)^{-1},$$

as a consequence of (5.5). We obtain the representation formula

$$\lim_{\alpha\to 0}\big\langle g(u)\big\rangle_\alpha \;=\; \int_{\mathbb{R}_0^+} g(s)\,\mu(ds) + g_*\sigma_{con} \quad \text{for all } g \in C_\rho(\mathbb{R}_0^+)\,,$$

with $\sigma_{con} \equiv \hat{\mu}\big(\{\infty\}\big)\,\sigma$ and with probability measure

$$\mu(ds) \;\equiv\; \left(\int_{\mathbb{R}_0^+} \frac{\hat{\mu}(ds)}{1+|s|^\rho}\right)^{-1} \frac{\hat{\mu}|_{\mathbb{R}_0^+}(ds)}{1+|s|^\rho}\,.$$

Using this in (5.4), we conclude as in Section 4.4 above that

(1) $\sigma_{con} = 0$;
(2) the measure μ is concentrated at one point.

Let $\langle g\rangle \equiv \int_{\mathbb{R}_0^+} g(s)\,\mu(ds)$ for $g \in C_\rho(\mathbb{R}_0^+)$. Then (5.3), (5.6) yield

$$\zeta_t(0,0) \;=\; -\big\langle f(u)\big\rangle \;=\; -f\big(\langle u\rangle\big) \;=\; -f\big(\zeta_x(0,0)\big)\,. \tag{5.7}$$

Note that μ is, in fact, uniquely determined by this relation, and so the whole sequence $\{\hat{\mu}_\alpha\}_\alpha$ must converge. From (5.7) we conclude that h is a viscosity solution of $h_t + f(h_x) = 0$. The proof is complete. $\square$

5.2. **Initial data.** We start from the entropy estimate (2.6)

$$\int_{\mathbb{R}} u_\nu^p(t)\,dx \;\le\; \int_{\mathbb{R}} \bar{u}^p\,dx \quad \text{for a.a. } t \in \mathbb{R}_0^+\,,$$

which is true for any admissible solution u_ν of (2.1). Then lower semicontinuity implies that a similar estimate holds for the limit $\hat{u}$, and

$$\limsup_{t\to 0} \big\|\hat{u}(t)\big\|_{L^p(\mathbb{R})} \;\le\; \|\bar{u}\|_{L^p(\mathbb{R})}\,. \tag{5.8}$$

On the other hand, for any admissible solution u_ν of (2.1) we have

$$\begin{aligned}
&\iint_{\mathbb{R}_0^+\times\mathbb{R}} \Big\{ u_\nu\,\varphi_t\xi + f(u_\nu)\,\varphi\xi_x \Big\}\,dx\,dt + \int_{\mathbb{R}} \bar{u}\,\varphi(0)\xi\,dx \\
&\qquad + \varepsilon \iint_{\mathbb{R}_0^+\times\mathbb{R}} \Big\{ -\tfrac{1}{n+1} u_\nu^{n+1}\,\xi_{xxxx} + \tfrac{3n}{2} u_\nu^{n-1} (u_\nu)_x^2\,\xi_{xx} \\
&\qquad\qquad + \tfrac{n(n-1)}{2} u_\nu^{n-2} (u_\nu)_x^3\,\xi_x \Big\} \varphi\,dx\,dt \;=\; 0
\end{aligned}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}_0^+)$ and $\xi \in \mathcal{D}(\mathbb{R})$. Passing to the limit $\nu \to \infty$ gives

$$\iint_{\mathbb{R}_0^+\times\mathbb{R}} \Big\{ \hat{u}\,\varphi_t\xi + f(\hat{u})\,\varphi\xi_x \Big\}\,dx\,dt + \int_{\mathbb{R}} \bar{u}\,\varphi(0)\xi\,dx \;=\; 0\,,$$

using Proposition 4.4 and Lemma 4.3. Fix some $T > 0$ and consider a sequence $\{\varphi_m\}_m$ with $\varphi_m \uparrow \mathbf{1}_{[0,T)}$ a.e. For almost every choice of T

$$\lim_{m\to\infty} \int_{\mathbb{R}_0^+\times\mathbb{R}} \hat{u}\,(\varphi_m)_t\xi\,dx\,dt \;=\; -\int_{\mathbb{R}} \hat{u}(T)\,\xi\,dx\,.$$

Therefore there exists a null set $E \subset \mathbb{R}_0^+$ such that for all $T \in \mathbb{R}_0^+\backslash E$

$$\int_{\mathbb{R}} \Big(\hat{u}(T) - \bar{u}\Big)\,\xi\,dx \;=\; \int_0^T \int_{\mathbb{R}} f(\hat{u})\,\xi_x\,dt \quad \text{for all } \xi \in \mathcal{D}(\mathbb{R})\,.$$

Since f has growth strictly less than cubic and $\hat{u} \in L^3(\mathbb{R}_0^+ \times \mathbb{R})$, this identity implies that $\operatorname{ess\,lim}_{t\to 0} \hat{u}(t) = \bar{u}$ in distributional sense. Then the uniform bound (5.8) allows to improve convergence to

$$\lim_{k\to\infty} \hat{u}(t_k) \;=\; \bar{u} \quad \text{weakly in } L^p(\mathbb{R})\,,$$

for any sequence $t_k \to 0$ with $t_k \in \mathbb{R}_0^+\backslash E$. Lower semicontinuity gives

$$\|\bar{u}\|_{L^p(\mathbb{R})} \;\le\; \liminf_{k\to\infty} \big\|\hat{u}(t_k)\big\|_{L^p(\mathbb{R})}\,,$$

which together with (5.8) implies convergence of the norms. Then (2.11) follow from standard functional analysis arguments.

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