SOME NEW VELOCITY AVERAGING RESULTS

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Abstract. Let (R, μ) be a nonatomic finite measure space and $E = L^{\underline{r}}(R, \mu)$ a Lebesgue space over R. Then we consider tempered distributions f and g (depending on $x \in \mathbb{R}^n$ and $v \in R$), for which $\mathrm{div}_x(\mathbf{a}f) = g$ in $\mathcal{S}'(\mathbb{R}^n, E)$. Here $\mathbf{a} : R \longrightarrow \mathbb{R}^n$ is a bounded function of v (a velocity field) satisfying a nondegeneracy condition. We study the regularity of the average $\bar{f} = \int_R f(\cdot, v)\psi(v)\, d\mu(v) \in \mathcal{S}'(\mathbb{R}^n)$ (with $\psi \in L^{\underline{r}'}(R, \mu)$ a suitable weight function) when f and g are bounded in Banach space valued Besov spaces. We also present some compactness results for sequences of averages.

Key words. Kinetic Equations, Velocity Averaging, Regularity, Compactness

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1. Introduction. The subject of velocity averaging is the regularity of moments of solutions of transport equations. Let us consider a typical situation: Assume functions f and g are given, depending on $x \in \mathbb{R}^n$ (space) and $v \in \mathbb{R}^n$ (velocity), for which the relation

$$v \cdot \nabla_x f = g \quad \text{in } \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n) \tag{1.1}$$

holds. Assume further that we know the regularity of the two functions, e.g. f and g bounded in $L_p(\mathbb{R}^n \times \mathbb{R}^n)$. What can be said about the velocity average

$$\bar{f}(x) = \int_{\mathbb{R}^n} f(x, v)\psi(v)\, dv?$$

Here $\psi \in \mathcal{D}(\mathbb{R}^n)$ is a suitable weight function. It turns out that $\bar{f}$ is somewhat smoother than f and g. Agoshkov [1] showed that if f and g are in $L_2(\mathbb{R}^n \times \mathbb{R}^n)$ and if the weight function is chosen suitably, then the average is bounded in the Sobolev space $W^{1/2,2}(\mathbb{R}^n)$. Hence we have a gain of one half derivative here. Golse, Lions, Perthame & Sentis [13] proved that $\bar{f} \in W^{1/2,2}(\mathbb{R}^n)$ for all $\psi \in \mathcal{D}(\mathbb{R}^n)$. Their proof is based on a v-dependent decomposition of the Fourier space. Using interpolation the authors also obtain a result for $1 < p < \infty$: If $f, g \in L_p(\mathbb{R}^n \times \mathbb{R}^n)$, then $\bar{f} \in W^{s,p}(\mathbb{R}^n)$ for all s strictly less than $\min\{1/p, 1/p'\}$.

DiPerna, Lions & Meyer [10] gave a further improvement. They proved that $\bar{f} \in B^s_{p,\max\{p,2\}}(\mathbb{R}^n)$ with $s = \min\{1/p, 1/p'\}$. Here $B^s_{p,q}(\mathbb{R}^n)$ is a Besov space (cf. §3.3). Bézard [3] showed that for $1 < p \leq 2$ the average is contained in the (slightly smaller) generalized Sobolev space $H^s_p(\mathbb{R}^n)$. Finally, DeVore & Petrova [7] made clear that $\bar{f} \in B^s_{p,p}(\mathbb{R}^n)$. They also proved that no further improvement w.r.t. the secondary index q of the Besov norm is possible.

There are several generalizations to the results given. One can assume different integrability for f and g, i.e. $f \in L_p(\mathbb{R}^n \times \mathbb{R}^n)$ and $g \in L_q(\mathbb{R}^n \times \mathbb{R}^n)$ for suitable p, q. DiPerna, Lions & Meyer [10] show for this case, that the average is contained in a Besov space built on Lorentz spaces instead of $L_p(\mathbb{R}^n)$-spaces as usual. Bézard [3] claims that $\bar{f}$ is even contained in some Sobolev space $H^s_r(\mathbb{R}^n)$, but there is a mistake in his proof.

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One can consider also the situation $f, g \in L_q(\mathbb{R}^n_v, L_p(\mathbb{R}^n_x))$ with $1 < q \leq p$. Then it is shown in [10], that $\bar{f} \in B^s_{p,t}(\mathbb{R}^n)$ with $s = \min\{1/q, 1/q'\}$ and suitable t. Note that s depends only on q, and that the integrability of $\bar{f}$ is the same as that of f und g, namely $L_p(\mathbb{R}^n)$. Bézard [3] also studies this situation for $q \leq 2$ and claims that $\bar{f} \in H^s_q(\mathbb{R}^n)$ with $s = 1/p'$, i.e. the roles of p and q are exchanged. But his proof is incorrect.
One can admit derivatives in v or even in x on the RHS of the transport equation. This was first studied by DiPerna & Lions [9]. But consult also DiPerna, Lions & Meyer [10]. More general transport operators such as the relativistic streaming operator or transport equations arising from a kinetic formulation of scalar conservation laws can also be considered, cf. Golse, Lions, Perthame & Sentis [13], DiPerna & Lions [8] and [9], Gérard [12], DiPerna, Lions & Meyer [10], Lions, Perthame & Tadmor [16].
The velocity averaging technique can also be used to study compactness. Assume sequences $f^{(k)}, g^{(k)}$ are given that satisfy the transport equation (1.1) and are uniformly bounded or precompact in some function space, what can be said about the convergence of the sequence $\bar{f}^{(k)}$? It is clear that the regularity results given above imply compactness. But we refer also to Golse, Perthame & Sentis [14], Golse, Lions, Perthame & Sentis [13], DiPerna & Lions [8], Lions, Perthame & Tadmor [16], Perthame & Souganidis [18], Bouchut [5] and Westdickenberg & Noelle [23].
Finally, let us mention that there is also a relationship between velocity averaging and results known as moments lemmata or dispersion lemmata. Here one considers solutions of the free transport equation $\partial_t f + v \cdot \nabla_x f = 0$ for suitable initial data $f(0, \cdot) = f_0$ with given integrability w.r.t. x and v. Then one asks what integrability f has in t, x and v. In particular, one is interested to find decay estimates for f in time. We refer to Perthame [17], Castella and Perthame [6], Bouchut [5] and the references given there.
In this paper, we present some new velocity averaging results. The starting point of our investigation was the question whether one can gain more than half a derivative in regularity by assuming more integrability in the kinetic variable v. In a sense, our results give an affirmative answer to this question, cf. §2. We were also interested to find out what are the weakest assumptions on f and g that would still guarantee strong precompactness. We give some answers to that question below.

This paper is organized as follows: In §2 we first develop a new view on velocity averaging and state our regularity and compactness results. In §3 we then collect some facts from the theory of Banach space valued tempered distributions. Sections 4 and 5 contain the proofs of our results.
We will assume in the following the space dimension $n \geq 2$. We denote by $\mathcal{D}(\mathbb{R}^n)$ the space of C^∞-functions with compact support, equipped with the usual topology of test functions. $\mathcal{D}'(\mathbb{R}^n)$ is the corresponding dual, the space of distributions.

2. Main Results. We want to make an attempt here to develop a somewhat different, less pragmatic view on velocity averaging than is usual in the literature. Therefore we go back to the transport equation $\operatorname{div}_x(\mathbf{a}f) = g$ and make precise in what sense this equation should hold. We will use notions and results from the theory of Banach space valued tempered distributions. The reader is referred to §3, where we put together some information relevant for our discussion.

Let (R, μ) be a nonatomic finite measure space and $E = L_{\underline{r}}(R, \mu), 1 \leq \underline{r} \leq \infty$ a Lebesgue space over (R, μ). Then we consider distributions f and g in $\mathcal{S}'(\mathbb{R}^n, E)$, i.e. linear mappings of the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ (x-dependance) into E (v-dependance), that are continuous w.r.t. to the Fréchet topology of $\mathcal{S}(\mathbb{R}^n)$. We assume that a function $\mathbf{a}\colon R \longrightarrow \mathbb{R}^n$ (velocity field) in $L_\infty(R,\mu)$ is given, and that the following relation holds for f and g

$$\operatorname{div}_x(\mathbf{a}f) = g \qquad \text{in } \mathcal{S}'(\mathbb{R}^n, E). \tag{2.1}$$

By the definition of $\mathcal{S}'(\mathbb{R}^n, E)$ this means that for all test functions $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$-\sum_{j=1}^{n} \mathbf{a}_j \langle f, \partial_j\varphi\rangle = \langle g, \varphi\rangle \qquad \text{in } E, \tag{2.2}$$

i.e. μ-almost everywhere (μ-a.e.). We use brackets to denote the dual pairing of distributions and test functions. Note that multiplication with $\mathbf{a} \in L_\infty(R,\mu)$ maps E continuously into itself. If both f and g are regular, we may also write

$$-\sum_{j=1}^{n} \mathbf{a}_j \int_{\mathbb{R}^n} f(x,\cdot)\partial_j\varphi(x)\,dx = \int_{\mathbb{R}^n} g(x,\cdot)\varphi(x)\,dx \qquad \text{in } E.$$

Now let ψ be an element of the conjugate space $E' = L_{\underline{r}'}(R,\mu)$ with $1/\underline{r} + 1/\underline{r}' = 1$. Then we can define the average $\bar{f}$ to be that distribution in $\mathcal{S}'(\mathbb{R}^n)$ for which

$$\langle \bar{f}, \varphi\rangle = \int_R \langle f(\cdot, v), \varphi\rangle \psi(v)\,d\mu(v), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{2.3}$$

Hence the average is the pairing of $f \in \mathcal{S}'(\mathbb{R}^n, E)$ with some $\psi \in E'$. Assume now that boundedness of f and g in suitable function spaces is given. Then we may ask for the regularity of $\bar{f}$.

2.1. Regularity. We will use Banach space valued Besov spaces $B^s_{p,q}(\mathbb{R}^n, E)$ with $s \in \mathbb{R}$ and $0 < p, q \le \infty$. Assume that we are given two tempered distributions

$$f \in B^s_{p,q}(\mathbb{R}^n, E_1) \quad \text{and} \quad g \in B^{s-\tau}_{p,q}(\mathbb{R}^n, E_2)$$

for suitable numbers $s, \tau \in \mathbb{R}$ and spaces $E_1 = L_{r_1}(R,\mu)$ and $E_2 = L_{r_2}(R,\mu)$. We will consider two different cases: $0 < p \le 1$ (Case I) and $1 < p < \frac{n}{n-1}$ (Case II). We put $E = L_{\underline{r}}(R,\mu)$ with $\underline{r} = \min\{r_1, r_2\}$ and assume that (2.1) holds.
Now let $F = L_r(R,\mu)$ be another Lebesgue space with $1/r \le \min\{1/r_1', 1/r_2'\}$. Then multiplication by some $\psi \in F$ maps E_1 continuously into $L_{\rho_1}(R,\mu)$ where $1/\rho_1 = 1/r + 1/r_1$, and analogously E_2 into some $L_{\rho_2}(R,\mu)$. We fix a weight $\psi \in F$ and define the average $\bar{f}$ as in (2.3).
We will assume further that the velocity field $\mathbf{a}$ is nondegenerate in the following sense: There are numbers $C > 0$ and $\alpha \in (0,1]$ s.t. for all $\delta \in (0,1]$

$$\sup_{\xi\in\mathbb{R}^n} \mu\bigl\{v \in R\colon \bigl|\mathbf{a}(v)\cdot\xi/|\xi|\bigr| \le \delta\bigr\} \le C\delta^\alpha. \tag{2.4}$$

Remark 2.1. *This condition was first used in [16]. Let us give an example. If R is a compact subset of $\mathbb{R}^n$ and μ the Lebesgue measure, then for $\mathbf{a}(v) = v$ (2.4) is satisfied with $\alpha = 1$. This velocity field appears e.g. in the Boltzmann Equations.*
Theorem 2.2. *Let $0 < q \le \infty, 0 < \tau \le 1$ and $\alpha/\rho_2' < 1$. With the assumptions above, $\bar{f}$ is bounded in $B^S_{P,q}(\mathbb{R}^n)$ for numbers P and $S = s - \kappa + \Delta S$ given by*

Case I	$0 < p \leq 1$	$P = 2$	$\kappa = n\Big(\frac{1}{p} - \frac{1}{2}\Big)$
Case II	$1 < p < \dfrac{n}{n-1}$	$P = \Big[\dfrac{1}{p} - \dfrac{n-1}{n}\Big]^{-1}$	$\kappa = n - 1$

and

$$\Delta S = (1-\tau)\frac{\alpha}{\rho_1'}\left[1 + \alpha\Big(\frac{1}{\rho_2} - \frac{1}{\rho_1}\Big)\right]^{-1}. \tag{2.5}$$

More precisely, there exists some constant $C > 0$ s.t. for all f, g and ψ in the respective function spaces the following inequality holds

$$\|\bar{f}\|_{B^S_{P,q}(\mathbb{R}^n)} \leq C\|\psi\|_F\Big\{\|f\|_{B^s_{p,q}(\mathbb{R}^n,E_1)} + \|g\|_{B^{s-\tau}_{p,q}(\mathbb{R}^n,E_2)}\Big\}, \tag{2.6}$$

whenever f and g satisfy the transport equation (2.1).

Remark 2.3. *The regularity of the average differs from that of f (i.e. s) by two terms. First, we loose $n(\frac{1}{p} - \frac{1}{P})$ derivatives. More precisely, we change regularity for integrability: Instead of $L_p(\mathbb{R}^n)$-boundedness we now have $L_P(\mathbb{R}^n)$. This is simply a Sobolev imbedding. Second, we gain some regularity ΔS which is a nonnegative number. This regularizing effect is an outcome of the nondegeneracy of* $\mathbf{a}$. *Note that ΔS does not depend on p and q. It is a function of α, τ, ρ_1 and ρ_2 only.*

Let us discuss a few special cases. If $\alpha = 1$, $\tau = 0$ and $r = \infty$, then $\Delta S \to 1$ for r_1 and r_2 getting large. In other words, we gain almost a full derivative if we have much integrability w.r.t. the kinetic variable v (remember that (R,μ) is a finite measure space). For $r_1 = r_2 = 2$ we find $\Delta S = 1/2$. Suppose now that g is a full derivative less regular than f, i.e. $\tau = 1$, then $\Delta S = 0$. This is obvious since in that case the transport equation contains no nontrivial regularity information. Note also that ΔS is getting smaller if r is chosen small: For more general (less integrable) weights $\psi \in L_r(R,\mu)$ we pay with a smaller gain of regularity.

Here is an example: If we consider the transport equation $v \cdot \nabla_x f = g$, for which the nondegeneracy condition (2.4) holds with $\alpha = 1$, and if both f and g are contained in $B^0_{1,1}(\mathbb{R}^n, L_\infty(R,\mu))$, then for $\psi \in L_\infty(R,\mu)$ the average $\bar{f}$ is in the negative Sobolev space $H^\epsilon_2(\mathbb{R}^n)$ for all $\epsilon < -\frac{n}{2} + 1$. In the particular case $n = 2$, this means that the average is almost in $L_2(\mathbb{R}^n)$. We cannot reach $\epsilon = 0$ because α/ρ_2' has to be strictly less than 1 (i.e. although the velocity field $\mathbf{a}$ *would admit $\alpha = 1$, in our estimates we have to use an $\alpha < 1$). This is an improvement over the results mentioned in §1 since there at most quadratical integrability in v could be handled, giving a maximal gain of regularity of one half derivative. Note that for an arbitrary Banach space E the following embedding holds*

$$B^0_{1,1}(\mathbb{R}^n, E) \hookrightarrow L_1(\mathbb{R}^n, E) \hookrightarrow B^0_{1,\infty}(\mathbb{R}^n, E) \tag{2.7}$$

(follows immediately from the definitions). Therefore, if we start with f and g in $L_1(\mathbb{R}^n, L_\infty(R,\mu))$, we obtain $\bar{f} \in B^\epsilon_{2,\infty}(\mathbb{R}^n)$ with ϵ as above. The latter space is slightly larger than the corresponding Sobolev space $H^\epsilon_2(\mathbb{R}^n)$.

Remark 2.4. *Clearly, it would be nice to get rid of the Sobolev embedding, at least partially. And it is also a little bit disappointing that the two cases do not match for $p = 1$ unless $n = 2$: The P for $p > 1$ is considerably larger than that for $p \leq 1$ (hence we loose more derivatives in the Sobolev embedding). The reason for this discrepancy is that we employ different methods of proof for the two cases of our theorem: for the* *first case we use a decomposition of f and g into simple building blocks which gives a decoupling of the x- and v-dependance; for the second case we use the Radon transform and its regularizing properties (cf. §4). We are not aware of a straightforward way to connect these two methods and thereby fill the gap in P mentioned above. Nevertheless, we hope that our methods are interesting for their own sake.*

2.2. Compactness. We now want to discuss a few compactness results and start with a generalization of what was said in §2.1. Let (R,μ) again be a nonatomic finite measure space, and let E_1 and E_2 be two arbitrary rearrangement-invariant Banach function spaces (cf. §5.1). Then we consider sequences of distributions bounded in Banach space valued Besov spaces

$$f^{(k)} \in B^s_{p,q}(\mathbb{R}^n, E_1) \quad \text{and} \quad g^{(k)} \in B^{s-\tau}_{p,q}(\mathbb{R}^n, E_2), \tag{2.8}$$

that satisfy the transport equation (2.1) in $\mathcal{S}'(\mathbb{R}^n, E)$ with $E = E_1 + E_2$. Assume there exists a subset F of the associated space E' of E and two further Banach function spaces G_1 and G_2 such that multiplication with $\psi \in F$ maps E_1 continuously into G_1, and E_2 into G_2. We are interested in precompactness of the sequence of averages $\bar{f}^{(k)}$ (cf. definition (2.3)) in local Besov spaces $B^{S,loc}_{P,q}(\mathbb{R}^n)$.
More precisely, we want to identify circumstances under which the sequence of products $\chi\bar{f}^{(k)}$ contains a subsequence converging in $B^S_{P,q}(\mathbb{R}^n)$, where $\chi \in \mathcal{D}(\mathbb{R}^n)$ is an arbitrary test function with compact support. Again we must assume nondegeneracy of the velocity field $\mathbf{a}$ which now takes the form

$$\lim_{\delta\to 0} \eta(\delta) = 0, \qquad \text{where } \eta(\delta) = \sup_{\xi\in\mathbb{R}^n} \mu\big\{v \in R\colon \big|\mathbf{a}(v)\cdot\xi/|\xi|\big| \le \delta\big\}. \tag{2.9}$$

Note that this assumption is weaker than condition (2.4). Again we consider two different situations: $0 < p \le 1$ and $1 < p < \frac{n}{n-1}$. Then we have
THEOREM 2.5. *Let $0 < q \le \infty$ and $0 < \tau < 1$. Assume that the fundamental function of the associated space G_1' of G_1 is continuous at zero. Then the sequence of averages $\bar{f}^{(k)}$ is precompact in $B^{S,loc}_{P,q}(\mathbb{R}^n)$, where $S = s_1 - \kappa$ with*

Case I	$0 < p \le 1$	$P = 2$	$\kappa = n\Big(\frac{1}{p} - \frac{1}{2}\Big)$
Case II	$1 < p < \dfrac{n}{n-1}$	$P = \Big[\dfrac{1}{p} - \dfrac{n-1}{n}\Big]^{-1}$	$\kappa = n-1$

If $\tau = 1$ we still have precompactness if we assume that the sequence $g^{(k)}$ is not only bounded in $B^{s-\tau}_{p,q}(\mathbb{R}^n, E_2)$, but strongly precompact.
For the definition of the fundamental function we refer to §5.1.
REMARK 2.6. *The gain of regularity due to the averaging process depends primarily on the nondegeneracy of the velocity field $\mathbf{a}$ and the integrability of f w.r.t. the kinetic variable v: the higher α and ρ_1 are in Theorem 2.2, the bigger ΔS becomes. Here we consider a situation where $\mathbf{a}$ is only weakly nondegenerate (i.e. we only assume that $\eta(\delta) \to 0$ for $\delta \to 0$; nothing is said about a polynomial rate) and where also the v-integrability of $f^{(k)}$ is only slightly better than $L_1(R,\mu)$. Then we still have precompactness, but there is no gain of regularity at all. Of course, if we strengthen our assumptions, we will get more: If instead of (2.9) we have (2.4), and if E_1, E_2 are Lebesgue spaces as above, then we can combine the proofs of Theorem 2.2 and Theorem 2.5 to show precompactness in $B^{S,loc}_{P,q}(\mathbb{R}^n)$ with regularity S strictly less then $s - n(\frac{1}{p} - \frac{1}{P}) + \Delta S$ and ΔS given by (2.5).*

Remark 2.7. *Let us discuss a few examples. The choice $E_1 = L^1(R,\mu)$ forces $F = L^\infty(R,\mu)$ and hence $G_1 = L^1(R,\mu)$ because $L^1(R,\mu)$ is the largest of all rearrangement-invariant Banach function spaces over (R,μ). The fundamental function of the corresponding associated space $G_1' = L^\infty(R,\mu)$ is discontinuous at zero (cf. §5.1). Therefore, Theorem 2.5 does not apply: You need more than simple L^1-integrability in the kinetic variable v to obtain strong compactness.*
A sufficient condition would be for example $E_1 = L\log L(R,\mu)$ and $F = L^\infty(R,\mu)$. Then $G_1 = L\log L(R,\mu)$ and the fundamental function of $G_1' = \exp L(R,\mu)$ is continuous at zero. $L\log L$-integrability plays an important role for the Boltzmann Equations: If f is a solution of this system, then $f\log f$ is the entropy density, and the famous H-Theorem tells us that the global entropy does not increase in time.
Note finally, that the choice $F = E_1'$ does not work either, since from the Hölder Inequality (5.1) we again obtain only L^1-integrability for the product.
Concretely: If $f^{(k)}$ and $g^{(k)}$ are bounded in the Besov spaces $B^0_{1,1}(\mathbb{R}^n, L\log L(R,\mu))$ and $B^{-\tau}_{1,1}(\mathbb{R}^n, L_1(R,\mu))$ with $\tau < 1$, then for $\psi \in L_\infty(R,\mu)$ the average $\bar{f}^{(k)}$ is locally precompact in the Sobolev space $H^\epsilon_2(\mathbb{R}^n)$ with $\epsilon = -\frac{n}{2}$. We can admit $\tau = 1$, if we assume strong precompactness instead of mere boundedness for $g^{(k)}$. If $f^{(k)}$ is only bounded in $L_1(\mathbb{R}^n, L\log L(R,\mu))$ instead, then we can use the embedding (2.7) to obtain local precompactness of $\bar{f}^{(k)}$ in $B^\epsilon_{2,\infty}(\mathbb{R}^n)$ with ϵ as above.
Remark 2.8. *To prove precompactness of $\bar{f}^{(k)}$ it is sufficient to have boundedness of $f^{(k)}$ and $g^{(k)}$ in local Besov spaces only, cf. Remark 5.9 below.*

3. Preliminaries. We collect here some results we will need in the proofs later on. We start with a few remarks about Banach space valued distributions.

3.1. Banach space valued tempered distributions. If E is some arbitrary Banach space, we define the Schwartz class $\mathcal{S}(\mathbb{R}^n, E)$ to be the space of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}^n$ taking their values in E. This space is locally convex and complete with respect to the Fréchet topology defined by the family of seminorms

$$p_N(\varphi) = \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} (1+|x|)^N \|\partial^\alpha\varphi(x)\|_E \quad \text{with } N \in \mathbb{N}_0.$$

We abbreviate $\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n, \mathbb{C})$. Now let $\mathcal{O}_M(\mathbb{R}^n, E)$ be the space of smooth E-valued functions with at most polynomial growth at infinity (also called slowly increasing). Again the topology is defined by a family of seminorms

$$\psi \longmapsto \sup_{x\in\mathbb{R}^n} \|\varphi(x)\partial^\alpha\psi(x)\|_E, \quad \text{for } \varphi \in \mathcal{S}(\mathbb{R}^n), \alpha \in \mathbb{N}_0^n. \tag{3.1}$$

We denote by $\mathcal{S}'(\mathbb{R}^n, E)$ the space of linear mappings from $\mathcal{S}(\mathbb{R}^n)$ into E, that are continuous w.r.t. the strong topology of the Schwartz class. The dual pairing of some $f \in \mathcal{S}'(\mathbb{R}^n, E)$ with a test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is expressed using brackets: $\langle f, \varphi\rangle$. Note that this quantity is an element of E. Therefore equality in $\mathcal{S}'(\mathbb{R}^n, E)$ means equality in E after testing against $\varphi \in \mathcal{S}(\mathbb{R}^n)$. If $f \in L^1_{loc}(\mathbb{R}^n, E)$, we can realize the pairing as an integral. As we did for $\mathcal{S}(\mathbb{R}^n)$, we will simply write $\mathcal{O}_M(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ whenever $E = \mathbb{C}$.
Exactly as in the scalar case we can define derivatives of E-valued tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n, E)$ or the product of f with a slowly increasing function in $\mathcal{O}_M(\mathbb{R}^n)$. Also the notions of support $\operatorname{supp} f$ of f, Fourier transform $\hat{f} = \mathbf{F}f$ and its inverse $\check{f} = \mathbf{F}^{-1}f$, and convolution $f \star \rho$ for $\rho \in \mathcal{S}(\mathbb{R}^n)$ can be carried over from the scalar

theory. Again we have the identity $f \star \rho = \mathbf{F}^{-1}[\hat{\rho}\mathbf{F}f]$ in $\mathcal{S}'(\mathbb{R}^n, E)$. Instead of going into details here, we refer to Amann [2] or Hörmander [15].

Let us assume now that besides E there exist two more Banach spaces F and G, and a bilinear continuous mapping $\cdot\colon F \times E \longrightarrow G$ with norm not bigger than one. We call this mapping a multiplication. Then we can define the product $a \bullet f$ of some E-valued tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n, E)$ with a function $a = \psi \otimes \chi$, where $\psi \in \mathcal{O}_M(\mathbb{R}^n)$ and $\chi \in F$, to be that distribution in $\mathcal{S}'(\mathbb{R}^n, G)$, for which

$$\big\langle a \bullet f, \varphi \big\rangle = \chi \cdot \langle f, \psi\varphi \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{3.2}$$

Note that $\langle f, \psi\varphi \rangle \in E$. Then we can ask whether that definition can be extended to a class of functions more general than $a = \psi \otimes \chi$.

THEOREM 3.1. *There exists a uniquely defined bilinear mapping*

$$\begin{aligned} \bullet\colon \mathcal{O}_M(\mathbb{R}^n, F) \times \mathcal{S}'(\mathbb{R}^n, E) &\longrightarrow \mathcal{S}'(\mathbb{R}^n, G) \\ (a, f) &\longmapsto a \bullet f \end{aligned}$$

with (3.2) for all $a = \psi \otimes \chi$. The mapping $\bullet$ is uniformly continuous w.r.t. each variable if the respective other one is confined to bounded subsets.

REMARK 3.2. *This is a special case of the Schwartz Kernel Theorem, cf. Theorem 2.1 in Amann [2]. The idea of the proof is the following: Choose some test function $\eta \in \mathcal{D}(\mathbb{R}^n)$ with $\eta \geq 0$ and $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$. For $\epsilon > 0$ define*

$$\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon), \quad \forall x \in \mathbb{R}^n.$$

Then the convolution $f_\epsilon = f \star \eta_\epsilon$ is a function in $\mathcal{O}_M(\mathbb{R}^n, E)$ with $\lim_{\epsilon\to 0} f_\epsilon = f$ in $\mathcal{S}'(\mathbb{R}^n, E)$. If $a \in \mathcal{O}_M(\mathbb{R}^n, F)$ is given, we can define the product $a \bullet f_\epsilon$ pointwise by $a \bullet f_\epsilon(x) = a(x) \cdot f_\epsilon(x) \in G$ for all $x \in \mathbb{R}^n$. Then we put

$$a \bullet f = \lim_{\epsilon \to 0} a \bullet f_\epsilon \quad \text{in } \mathcal{S}'(\mathbb{R}^n, G).$$

Of course, a little work is necessary to show that this definition makes sense. If there is no danger of confusion we will abbreviate $a \bullet f$ by af.

REMARK 3.3. *We also need the following fact. Assume there exist Banach spaces E, F_1, F_2, G, H_1, H_2 and multiplications (all denoted by $\cdot$)*

$$\begin{array}{ccccccccc} F_1 & \times & F_2 & & & & F_2 & \times & E \\ & \downarrow & & & & & & \downarrow & \\ & H_1 & \times & E & \longrightarrow & G & \longleftarrow & F_1 & \times & H_2 \end{array}$$

that are associative, i.e. $(\chi_1 \cdot \chi_2) \cdot e = \chi_1 \cdot (\chi_2 \cdot e)$ in G for all $\chi_j \in F_j$ and $e \in E$. Then the pointwise multiplication of Theorem 3.1 is associative as well. More precisely,

$$(M_1 \bullet M_2) \bullet f = M_1 \bullet (M_2 \bullet f) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, G)$$

for all $M_j \in \mathcal{O}_M(\mathbb{R}, F_j)$ and $f \in \mathcal{S}'(\mathbb{R}^n, E)$ (cf. Amann [2]).

3.2. L_p-Spaces of Banach space valued entire functions. Let $f \in \mathcal{S}'(\mathbb{R}^n, E)$ be a distribution with the property that the support of its Fourier transform $\mathbf{F}f$ is contained in a compact subset $K \subset \mathbb{R}^n$. If $\rho \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\hat{\rho}$ compact and $\hat{\rho}(\xi) = 1, \forall \xi \in K$, then the identity $f = \mathbf{F}^{-1}[\hat{\rho}\mathbf{F}f] = f \star \rho$ in $\mathcal{S}'(\mathbb{R}^n, E)$ follows immediately from the definitions. Now ρ is an entire analytic function on $\mathbb{R}^n$, that can be extended to $\mathbb{C}^n$. This is the famous Payley-Wiener-Schwartz Theorem. For any $N \in \mathbb{N}$ there exists some constant $C_N > 0$ s.t.

$$|\rho(z)| \le C_N(1+|z|)^{-N} e^{c|\mathrm{Im} z|}, \quad \forall z \in \mathbb{C}^n. \tag{3.3}$$

From this estimate one derives, completely analogous to the scalar valued case, that $f \in \mathcal{S}'(\mathbb{R}^n, E)$ with $\operatorname{supp} \mathbf{F} f$ compact is an entire analytic E-valued function, too. We refer to [15], Theorem 7.3.1 for the argument with $E = \mathbb{C}$.

Definition 3.4. *Let E be some Banach space, $K \subset \mathbb{R}^n$ compact and $0 < p \le \infty$. Then we define the L_p-space of E-valued entire analytic functions*

$$L_{p,K}(\mathbb{R}^n, E) = \Big\{ f \in \mathcal{S}'(\mathbb{R}^n, E) \colon \operatorname{supp} \hat{f} \subset K, \|f\|_{L_p(\mathbb{R}^n,E)} < \infty \Big\},$$
$$\text{with} \quad \|f\|_{L_p(\mathbb{R}^n,E)} = \left(\int_{\mathbb{R}^n} \|f(x)\|_E^p \, dx \right)^{1/p}.$$

The space $L_{p,K}(\mathbb{R}^n, E)$ is complete w.r.t. $\|\cdot\|_{L_p(\mathbb{R}^n,E)}$.

Theorem 3.5. *Let E be some Banach space, $K \subset \mathbb{R}^n$ compact, $0 < p \le \infty$ and $0 < w < \infty$. Then there exists a constant $C > 0$ s.t. for all $f \in L_{p,K}(\mathbb{R}^n, E)$*

$$\sup_{y \in \mathbb{R}^n} \|f(x-y)\|_E \big(1+|y|\big)^{-n/w} \le C\big(\mathbf{M}\|f\|_E^w\big)^{1/w}(x), \quad \forall x \in \mathbb{R}^n.$$

Here $\mathbf{M}$ is the usual Hardy-Littlewood maximal operator

$$\mathbf{M} g(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |g(y)| \, dy \colon \textit{all cubes } Q \textit{ containing } x \right\}.$$

If now $0 < w < p$ (hence $p/w > 1$), we obtain as an immediate consequence of the Hardy-Littlewood maximal inequality (cf. Stein [20])

$$\left\| \sup_{y \in \mathbb{R}^n} \frac{\|f(\cdot - y)\|_E}{\big(1+|y|\big)^{n/w}} \right\|_{L^p(\mathbb{R}^n)} \le C \big\| \mathbf{M}\|f\|_E^w \big\|_{L^{p/w}(\mathbb{R}^n)}^{1/w}$$
$$\le C \big\| \|f\|_E^w \big\|_{L^{p/w}(\mathbb{R}^n)}^{1/w} = C\|f\|_{L_p(\mathbb{R}^n,E)}.$$

The constant $C = C(n, p, K, w)$ does not depend on f.

Theorem 3.6 (Nikol'skij inequality). *Let E and K be as above, $0 < p \le q \le \infty$ and $\alpha \in \mathbb{N}_0^n$. Then there exists a constant $C > 0$ s.t. for all $f \in L_{p,K}(\mathbb{R}^n, E)$*

$$\|\partial^\alpha f\|_{L_q(\mathbb{R}^n,E)} \le C\|f\|_{L_p(\mathbb{R}^n,E)}. \tag{3.4}$$

We refer to Triebel [22], Chapter III/15. and to the literature cited there.

3.3. Banach space valued Besov spaces. Let φ_0 be a radially symmetric test function in $\mathcal{S}(\mathbb{R}^n)$ supported in $B_2(0) \subset \mathbb{R}^n$ with $\varphi_0(\xi) = 1, \forall |\xi| \le 1$. Then define $\varphi_1(\xi) = \varphi_0(2^{-1}\xi) - \varphi_0(\xi)$ and $\varphi_\nu(\xi) = \varphi_1(2^{-\nu+1}\xi)$ for $\nu \in \mathbb{N}$. We obtain a dyadic decomposition of unity $\sum_{\nu \in \mathbb{N}_0} \varphi_\nu(\xi) = 1, \forall \xi \in \mathbb{R}^n$.

Definition 3.7. *Let E be an arbitrary Banach space and $0 < p, q \le \infty, s \in \mathbb{R}$. Then the Banach space valued Besov spave $B^s_{p,q}(\mathbb{R}^n, E)$ is defined as the space of all E-valued distributions $f \in \mathcal{S}'(\mathbb{R}^n, E)$, for which the Besov (quasi)norm*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n,E)} = \big\| \big\{ 2^{\nu s} \|f_\nu\|_{L_p(\mathbb{R}^n,E)} \big\}_\nu \big\|_{\ell_q(\mathbb{N}_0)} \tag{3.5}$$

is finite. Here $f_\nu = \mathbf{F}^{-1}[\varphi_\nu \hat{f}]$ *in* $\mathcal{S}'(\mathbb{R}^n, E)$*. For* $1 \le p, q \le \infty$ *(3.5) is a norm.*

Remark 3.8. *We stress that the Banach space is completely arbitrary. Assumptions like UMD-Property (Unconditionality of Martingale Differences), separability or reflexivity are not necessary. If* $E = \mathbb{C}$ *we will simply write* $B^s_{p,q}(\mathbb{R}^n)$*.*

Remark 3.9. *Assume* $t \in \mathbb{R}$*,* $0 < r \le \infty$ *and let* F *be a second Banach space continuously embedded into* E*. As an immediate consequence of definition (3.5), the following inclusions hold*

$$\begin{aligned} &B^s_{p,q}(\mathbb{R}^n, E) \hookrightarrow B^s_{p,r}(\mathbb{R}^n, E) \quad \text{if } q \le r, \\ &B^s_{p,q}(\mathbb{R}^n, E) \hookrightarrow B^t_{p,q}(\mathbb{R}^n, E) \quad \text{if } t \le s \text{ and} \\ &B^s_{p,q}(\mathbb{R}^n, E) \hookrightarrow B^s_{p,q}(\mathbb{R}^n, F). \end{aligned}$$

For any $\sigma \in \mathbb{R}$ the operator $\mathcal{J}_\sigma$, defined by

$$\widehat{\mathcal{J}_\sigma \varphi}(\xi) = \left(1 + |\xi|^2\right)^{\sigma/2} \hat{\varphi}(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n,$$

maps the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ injectively onto itself. A posteriori then, the same is true for the space $\mathcal{S}'(\mathbb{R}^n, E)$ because the product of $\mathbf{F}f \in \mathcal{S}'(\mathbb{R}^n, E)$ with a function $(1 + |\cdot|^2)^{\sigma/2} \in \mathcal{O}_M(\mathbb{R}^n)$ is well defined. We have $\mathcal{J}_\sigma \circ \mathcal{J}_{-\sigma} = \mathrm{Id}$ in $\mathcal{S}'(\mathbb{R}^n, E)$.

Theorem 3.10. *For numbers* $s \in \mathbb{R}$ *and* $0 < p, q \le \infty$ *the two spaces* $B^{pq}_{s-\sigma}(\mathbb{R}^n, E)$ *and* $\mathcal{J}_\sigma B^s_{p,q}(\mathbb{R}^n, E) = \left\{\mathcal{J}_\sigma f \colon f \in B^s_{p,q}(\mathbb{R}^n, E)\right\}$ *coincide. The quantity*

$$\|f\|^*_{B^s_{p,q}(\mathbb{R}^n,E)} = \|\mathcal{J}_\sigma f\|_{B^{s-\sigma}_{p,q}(\mathbb{R}^n,E)}$$

is an equivalent (quasi)norm on $B^s_{p,q}(\mathbb{R}^n, E)$*.*

Remark 3.11. *So Besov spaces of different regularity (with the same* p, q *of course) are isomorphic to each other. The mapping* $\mathcal{J}_\sigma$ *is called a lifting. Let us also recall the closely related estimate*

$$\|\partial^\alpha f\|_{B^s_{p,q}(\mathbb{R}^n,E)} \le C \|f\|_{B^{s+|\alpha|}_{p,q}(\mathbb{R}^n,E)}, \quad \forall \alpha \in \mathbb{N}^n_0.$$

We will not give here the proof of Theorem 3.10 nor that of the next one, Theorem 3.12. In both cases, it is an easy adaption of the corresponding result for the scalar case. We refer to Triebel [21].

Theorem 3.12. *Let* $s \in \mathbb{R}$*,* $0 < p, q \le \infty$ *and* $\psi \in \mathcal{O}_M(\mathbb{R}^n)$*. Then for large* M

$$\|\psi f\|_{B^s_{p,q}(\mathbb{R}^n,E)} \le C \sum_{|\alpha| \le M} \|\partial^\alpha \psi\|_{L_\infty(\mathbb{R}^n)} \|f\|_{B^s_{p,q}(\mathbb{R}^n,E)}, \quad \forall f \in \mathcal{S}'(\mathbb{R}^n, E). \tag{3.6}$$

4. Proofs - Regularity. We briefly repeat the assumptions made in §2.1. Let (R, μ) be a nonatomic finite measure space and fix Lebesgue spaces $E_1 = L_{r_1}(R, \mu)$ and $E_2 = L_{r_2}(R, \mu)$ with $1 \le r_1, r_2 \le \infty$. Assume we are given $f \in B^s_{p,q}(\mathbb{R}^n, E_1)$ and $g \in B^{s-\tau}_{p,q}(\mathbb{R}^n, E_2)$ for some $0 < p, q \le \infty$ and $s, \tau \in \mathbb{R}$ satisfying a transport equation $\mathrm{div}_x(\mathbf{a}f) = g$ in $\mathcal{S}'(\mathbb{R}^n, E)$, where $E = L_{\underline{r}}(R, \mu)$ and $\underline{r} = \min\{r_1, r_2\}$. For an arbitrary $\psi \in F = L_r(R, \mu)$ with $1/r \le \min\{1/r'_1, 1/r'_2\}$ define the average $\bar{f} \in \mathcal{S}'(\mathbb{R}^n)$ by

$$\langle \bar{f}, \phi \rangle = \int_R \langle f(\cdot, v), \phi \rangle \psi(v)\, d\mu(v), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

We will prove that if the velocity field $\mathbf{a}$ satisfies a nondegeneracy condition (2.4), and provided that suitable assumptions on the various parameters hold (which we will repeat in the course of the proof as they are needed), then the average $\bar{f}$ is bounded in the Besov space $\bar{f} \in B^S_{P,q}(\mathbb{R}^n)$ for numbers P, S given in §2.1.
Without restriction of generality we may fix some s large because of a simple lifting argument: The operator $\mathcal{J}_\sigma$ introduced in §3.3 is an isomorphism between Banach space valued Besov spaces, $\mathcal{J}_\sigma B^s_{p,q}(\mathbb{R}^n, E) \approx B^{s-\sigma}_{p,q}(\mathbb{R}^n, E)$, and it commutes with the transport operator. We may therefore consider the distributions $F = \mathcal{J}_\sigma f$ resp. $G = \mathcal{J}_\sigma g$. They satisfy the same transport equation $\operatorname{div}_x(\mathbf{a}F) = G$. Then $\bar{F}$ is equal to $\mathcal{J}_\sigma \bar{f}$ which means that the regularity is simply shifted by σ.
The average $\bar{f}$ is a tempered distribution and can therefore be decomposed into its dyadic components $\bar{f} = \sum_{\nu=0}^\infty \bar{f}_\nu$ in $\mathcal{S}'(\mathbb{R}^n)$, with $\bar{f}_\nu = \mathbf{F}^{-1}[\varphi_\nu \mathbf{F} \bar{f}]$ as usual. It is then sufficient to estimate each block separately: We will show that there exists a constant $C > 0$ independent of f and g s.t. for all $\nu \geq 0$

$$(4.1) \qquad 2^{\nu S} \|\bar{f}_\nu\|_{L_P(\mathbb{R}^n)} \leq C \|\psi\|_F \left\{ 2^{\nu s} \|f_\nu\|_{L_p(\mathbb{R}^n, E_1)} + 2^{\nu(s-\tau)} \|g_\nu\|_{L_p(\mathbb{R}^n, E_2)} \right\}.$$

Here f_ν and g_ν are the dyadic components of f and g. We take the $\ell_q(\mathbb{N}_0)$-norm of the sequence $\left\{2^{\nu S} \|\bar{f}_\nu\|_{L_P(\mathbb{R}^n)}\right\}_{\nu=0}^\infty$, use the q-triangle inequality and are done.
To prove inequality (4.1) for $\nu = 0$ is a simple matter. Note that for all $\nu \geq 0$

$$\left\langle \bar{f}_\nu, \phi \right\rangle = \int_R \left\langle f_\nu(\cdot, v), \phi \right\rangle \psi(v) \, d\mu(v), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

This follows easily from the definitions. Now each f_ν is an entire analytic function because of the Payley-Wiener-Schwartz Theorem (cf. §3.1). We estimate

$$(4.2) \; \|\bar{f}_0\|_{L_P(\mathbb{R}^n)} = \left\| \int_R f_0(\cdot, v) \psi(v) \, d\mu(v) \right\|_{L_P(\mathbb{R}^n)} \leq \mu(R)^{1/\rho_1'} \|\psi\|_F \|f_0\|_{L_P(\mathbb{R}^n, E_1)}$$

and then use the Nikol'skij Inequality (3.4) with $\alpha = 0$ and $q = P$. Recall that R has finite μ-measure and that the exponent ρ_1 was defined by $1/\rho_1 = 1/r + 1/r_1$. This gives (4.1) for $\nu = 0$. So we will assume in the following that $\nu \geq 1$. Then both f_ν and g_ν are smooth functions, and the support of their Fourier transforms lies in a compact set strictly bounded away from the origin.
Since f and g satisfy the transport equation (2.1), the following identity holds

$$(4.3) \qquad \left(i\mathbf{a}(v) \cdot \xi\right) \hat{f}(\xi, v) = \hat{g}(\xi, v) \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E).$$

As explained in Theorem 3.1, the LHS must be understood as a product of the symbol $i\mathbf{a} \cdot \xi \in \mathcal{O}_M(\mathbb{R}^n, L_\infty(R, \mu))$ with the tempered distribution $\hat{f} \in \mathcal{S}'(\mathbb{R}^n, E)$. More precisely, the LHS is that distribution for which

$$\left\langle \left(i\mathbf{a} \cdot \xi\right) \hat{f}, \phi \right\rangle = \sum_{j=1}^n \mathbf{a}_j \left\langle \hat{f}, i\xi_j \phi \right\rangle, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n).$$

Note that multiplication with $\mathbf{a} \in L_\infty(R, \mu)$ maps E continuously into itself. Then identity (4.3) follows immediately from the definitions, and it still holds when we multiply both sides with φ_ν. So we may replace $\hat{f}$ and $\hat{g}$ with $\hat{f}_\nu$ and $\hat{g}_\nu$.
One might be tempted to divide by the the symbol $i\mathbf{a} \cdot \xi$ and express $\hat{f}_\nu$ by $\hat{g}_\nu$. However, products of distributions are defined only for smooth functions. And since

$(i\mathbf{a}\cdot\xi)^{-1}$ becomes unbounded for all $\xi\in\mathbb{R}^n$ orthogonal to $\mathbf{a}(v)$, we must be a little bit more careful. We introduce a splitting and use (4.3) only for that part of Fourier space, that is bounded away from the set of points where the symbol vanishes. This is the classical approach. We choose an arbitrary nonnegative test function $\Pi\in\mathcal{D}(\mathbb{R})$, vanishing outside the interval $[-1,1]$, with $\Pi(\xi)=1,\forall|\xi|\le\frac{1}{2}$. Then we can define functions

$$\chi_{\mathbf{s}}(\xi,v)=\Pi\big(\delta^{-1}\mathbf{a}(v)\cdot\xi/|\xi|\big)\quad\text{and}\quad\chi_{\mathbf{r}}(\xi,v)=1-\chi_{\mathbf{s}}(\xi,v)\tag{4.4}$$

for $(\xi,v)\in\mathbb{R}^n\times R$ and $\delta\ge 0$. Note that the support of $\chi_{\mathbf{s}}$ is contained in the set

$$A_\delta=\big\{(\xi,v)\in\mathbb{R}^n\times R\colon\big|\mathbf{a}(v)\cdot\xi/|\xi|\big|\le\delta\big\}.\tag{4.5}$$

We claim now that both $\varphi_\nu\chi_{\mathbf{s}}$ and $\varphi_\nu\chi_{\mathbf{r}}\big(i\mathbf{a}\cdot\xi\big)^{-1}$ are bounded in $\mathcal{O}_M(\mathbb{R}^n,L_\infty(R,\mu))$, for any $\nu\ge1$. If that is true, we obtain an identity

$$\hat f_\nu(\xi,v)=\chi_{\mathbf{s}}(\xi,v)\hat f_\nu(\xi,v)+\chi_{\mathbf{r}}(\xi,v)\frac{\hat g_\nu(\xi,v)}{i\mathbf{a}(v)\cdot\xi}\quad\text{in }\mathcal{S}'(\mathbb{R}^n,E),\tag{4.6}$$

for $\nu\ge1$. In fact, note that the second term on the RHS of (4.6) is well defined in $\mathcal{S}'(\mathbb{R}^n,E)$ because of Theorem 3.1. Then we may use the relation (4.3) together with Remark 3.3 to eliminate $\hat g_\nu$, and (4.6) follows from the definition of $\chi_{\mathbf{s}}$ and $\chi_{\mathbf{r}}$.
To prove our claim, let us start with $\chi_{\mathbf{s}}$. This function is homogeneous of degree zero in ξ. Hence ξ-derivatives of it of order k are homogeneous in ξ of degree $-k$ for $k\ge0$. More precisely, if $\alpha\in\mathbb{N}_0^n$ is some multiindex, $|\xi|^{|\alpha|}\partial_\xi^\alpha\chi_{\mathbf{s}}(\xi)$ is a linear combination of products of the following terms:

1. derivatives of Π taken at $\delta^{-1}\mathbf{a}(v)\cdot\xi/|\xi|$,
2. powers of $\delta^{-1}\mathbf{a}(v)\cdot\xi/|\xi|$ with positive exponent,
3. polynomials in $\mathbf{a}(v)\in\mathbb{R}^n$ and
4. polynomials in $\xi/|\xi|\in\mathbb{R}^n$.

This follows easily from an induction argument. We assumed that $\mathbf{a}\in L_\infty(R,\mu)$. Therefore all these terms are uniformly bounded w.r.t. v for any $\xi\in\mathbb{R}^n$ fixed. And since φ_ν vanishes in a neighborhood of zero for $\nu\ge1$, the functions $\varphi_\nu\chi_{\mathbf{s}}$ are bounded in $\mathcal{O}_M(\mathbb{R}^n,L_\infty(R,\mu))$. In the same manner, we can proceed for $\varphi_\nu\chi_{\mathbf{r}}$.
To prove that even $\varphi_\nu\chi_{\mathbf{r}}\big(i\mathbf{a}\cdot\xi\big)^{-1}$ is bounded in $\mathcal{O}_M(\mathbb{R}^n,L_\infty(R,\mu))$, note first that this function is homogeneous of degree -1 in ξ. Taking derivatives w.r.t. ξ we obtain the same terms we already had for $\chi_{\mathbf{s}}$, but now there are also powers of $\mathbf{a}(v)\cdot\xi/|\xi|$ with negative exponent. Still, these terms are uniformly bounded in v because $\chi_{\mathbf{r}}$ vanishes in $A_{\delta/2}$ by construction. This proves our claim.
Summing up we have a decomposition $\bar f_\nu=\bar f_{\mathbf{s},\nu}+\bar f_{\mathbf{r},\nu}$ for $\nu\ge1$ with

$$\big\langle\bar f_{\mathbf{s},\nu},\phi\big\rangle=\int_R\Big\langle\big[\chi_{\mathbf{s}}\hat f_\nu\big](\cdot,v),\check\phi\Big\rangle\psi(v)\,d\mu(v)\quad\text{and}\tag{4.7}$$

$$\big\langle\bar f_{\mathbf{r},\nu},\phi\big\rangle=\int_R\Big\langle\big[|\cdot|^{-1}\bar\chi_{\mathbf{r}}\hat g_\nu\big](\cdot,v),\check\phi\Big\rangle\psi(v)\,d\mu(v),\quad\forall\phi\in\mathcal{S}(\mathbb{R}^n).\tag{4.8}$$

We put $\bar\chi_{\mathbf{r}}(\xi,v)=\chi_{\mathbf{r}}(\xi,v)\big(i\mathbf{a}(v)\cdot\xi/|\xi|\big)^{-1}$. While up to now the argument was the same for both cases of our Theorem 2.2, we must now specialize a bit.

4.1. Regularity - Case I. We will use the fact that each $f\in B^s_{p,q}(\mathbb{R}^n,E)$ can be decomposed into an infinite sum of simple building blocks of the form $\lambda\mathbf{Q}$, where $\mathbf{Q}$ is a scalar function with nice properties and λ is an element of E.

4.1.1. The subatomic decomposition. Let $Q_{\nu m}$ be a cube in $\mathbb{R}^n$ centered at $2^{-\nu}m$ with sides parallel to the coordinate axes and side length $2^{-\nu}$, where $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $c > 1$ a number, we will write cQ for the cube concentric with Q, but with sides c times as long as those of Q. Now choose a nonnegative function $\psi \in \mathcal{S}(\mathbb{R}^n)$ with compact support in some cQ_{00} and

$$\sum_{m\in\mathbb{Z}^n} \psi(x-m) = 1 \qquad \text{for all } x \in \mathbb{R}^n. \tag{4.9}$$

Let $s \in \mathbb{R}, 0 < p \le \infty, \frac{L+1}{2} \in \mathbb{N}_0, \gamma \in \mathbb{N}_0^n$ and $\psi^\gamma(x) = x^\gamma \psi(x)$. Then the function

$$(\gamma qu)^L_{\nu m}(x) = 2^{-\nu(s-\frac{n}{p})}\Big((-\Delta)^{\frac{L+1}{2}}\psi^\gamma\Big)(2^\nu x - m) \tag{4.10}$$

is called an $(s,p)_L - \gamma$-quark over the cube $Q_{\nu m}$.

Remark 4.1. *This definition is taken from [22], Section 14.1. In the following, we will only need the case $L = -1$. Then the Laplace operator in (4.10) drops out.*

Before stating the subatomic decomposition for $B^s_{p,q}(\mathbb{R}^n, E)$, we show

Theorem 4.2. *Let $K \subset \mathbb{R}^n$ be a compact set and $0 < p \le \infty$. Then there exists a number $\kappa > 0$ with the following property: for any $\mu \in \mathbb{N}$ with $\mu > \kappa$ the entire analytic function $g \in L_{p,K}(\mathbb{R}^n, E)$ can be written as*

$$g(x) = \sum_{m\in\mathbb{Z}^n} \sum_{\gamma\in\mathbb{N}_0^n} \lambda^\gamma_m \psi^\gamma(2^\mu x - m) \tag{4.11}$$

where the sum converges in $L_q(\mathbb{R}^n, E)$ for all $q \in [p,\infty]$, with $\lambda^\gamma_m \in E$ s.t.

$$\sup_{\gamma\in\mathbb{N}_0^n} 2^{\mu|\gamma|} \left(\sum_{m\in\mathbb{Z}^n} \|\lambda^\gamma_m\|^p_E\right)^{1/p} \le C\|g\|_{L_p(\mathbb{R}^n,E)}. \tag{4.12}$$

The constant $C = C(K, \mu, n, p)$ does not depend on g.

Proof. Our proof simplifies a bit a similar argument in Triebel [22], Section 14.15. Choose $\rho \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\hat\rho$ contained in some compact neighborhood of K and $\hat\rho(\xi) = 1, \forall \xi \in K$. From the Paley-Wiener-Schwartz Theorem we infer that ρ is an entire analytic function. Moreover

$$g(x) = \int_{\mathbb{R}^n} g(y)\rho(x-y)\,dy, \quad \forall x \in \mathbb{R}^n \tag{4.13}$$

(cf. §3.2). Note that as a consequence of the Nikol'skij inequality (3.4) the function g is bounded, hence locally integrable. We now expand $\rho(\cdot - y)$ into a Taylor series around the point $2^{-\mu}m$ with $m \in \mathbb{Z}^n$ and $\mu \in \mathbb{N}$ fixed. Then

$$\begin{aligned}\psi(2^\mu x - m)\rho(x-y) &= \psi(2^\mu x - m) \sum_{\gamma\in\mathbb{N}_0^n} \frac{D^\gamma\rho(2^{-\mu}m - y)}{\gamma!}\big(x - 2^{-\mu}m\big)^\gamma \\ &= \sum_{\gamma\in\mathbb{N}_0^n} 2^{-\mu|\gamma|}\frac{D^\gamma\rho(2^{-\mu}m - y)}{\gamma!}\psi^\gamma(2^\mu x - m).\end{aligned} \tag{4.14}$$

Since $\operatorname{supp}\psi \subset cQ_{00}$ with $c > 1$, we have $|\psi^\gamma(2^\mu x - m)| \le \big(\frac{c}{2}\sqrt{2}^n\big)^{|\gamma|}$. We apply the Cauchy integral formula componentwise to ρ. Then we obtain for all $z \in \mathbb{C}^n$

$$\rho(z_1,\dots,z_n) = (2\pi i)^{-n}\int_{|w_1-z_1|=1}\cdots\int_{|w_n-z_n|=1} \frac{\rho(w_1,\dots,w_n)\,dw_1\cdots dw_n}{(z_1-w_1)\cdots(z_n-w_n)}$$

(integration over $\mathbb{C}^n$). Differentiation gives

$$D^\gamma \rho(z_1, \dots, z_n) \\ = (-1)^{|\gamma|}\gamma!(2\pi i)^{-n} \int_{|w_1 - z_1|=1} \cdots \int_{|w_n - z_n|=1} \frac{\rho(w_1, \dots, w_n)\, dw_1 \cdots dw_n}{(z_1 - w_1)^{\gamma_1+1} \cdots (z_n - w_n)^{\gamma_n+1}}$$

for $z \in \mathbb{C}^n$ and $\gamma \in \mathbb{N}_0^n$. Using (3.3) we now obtain for arbitrary $N \in \mathbb{N}$ the estimate

$$|D^\gamma \rho(z)| \leq C_N \gamma!(2\pi)^{-n} \int_{|w_1 - z_1|=1} \cdots \int_{|w_n - z_n|=1} (1 + |w|)^{-N} e^{c|\mathrm{Im} w|}\, dw_1 \dots dw_n.$$

If $z \in \mathbb{R}^n$, then $|\mathrm{Im} w| \leq 1$ in the domain of integration. Using

$$1 + |z| \leq (1 + |w|)(1 + |z - w|) \leq (1 + |w|)(1 + |z_1 - w_1| + \dots + |z_n - w_n|),$$

we can find some constant $C = C(K, N, n)$ s.t.

$$|D^\gamma \rho(z)| \leq C\gamma!(1 + |z|)^{-N} \quad \text{for all } z \in \mathbb{R}^n, \gamma \in \mathbb{N}_0^n. \tag{4.15}$$

The number of multi indices $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = k$ only grows polynomially in k. We conclude that the expansion (4.14) is absolutely convergent for μ large enough:

$$\left| \psi(2^\mu x - m)\rho(x - y) - \sum_{|\gamma| \leq K} 2^{-\mu|\gamma|} \frac{D^\gamma \rho(2^{-\mu} m - y)}{\gamma!} \psi^\gamma(2^\mu x - m) \right| \\ \leq C \mathbf{1}_{cQ_{\mu m}}(x) \big(1 + |2^{-\mu} m - y|\big)^{-N} \underbrace{\sum_{k=K+1}^{\infty} k^n 2^{-\mu k} \left(\frac{c}{2}\sqrt{2}^n\right)^k}_{c_K} \tag{4.16}$$

and $c_K \to 0$ for $K \to \infty$. This estimate implies (4.11) with

$$\lambda_m^\gamma = 2^{-\mu|\gamma|} \int_{\mathbb{R}^n} g(y) \frac{\big(D^\gamma \rho\big)(2^{-\mu} m - y)}{\gamma!} dy \in E. \tag{4.17}$$

To see that, note first that the sum in m does not cause any harm: For $x \in \mathbb{R}^n$ fixed only finitely many terms contribute to the sum because ψ is compactly supported. Using (4.9) we can write for any $x \in \mathbb{R}^n$

$$\left\| g(x) - \sum_{m \in \mathbb{Z}^n} \sum_{|\gamma| \leq K} \lambda_m^\gamma \psi^\gamma(2^\mu x - m) \right\|_E \\ \leq \sum_{m \in \mathbb{Z}^n} \left\| \psi(2^\mu x - m) g(x) - \sum_{|\gamma| \leq K} \lambda_m^\gamma \psi^\gamma(2^\mu x - m) \right\|_E .$$

Consider one single term of this sum. If $g(x)$ is replaced with (4.13) and λ_m^γ with (4.17), then we obtain the following

$$\left\| \int_{\mathbb{R}^n} g(y) \Bigg(\psi(2^\mu x - m)\rho(x - y) \\ - \sum_{|\gamma| \leq K} 2^{-\mu|\gamma|} \frac{\big(D^\gamma \rho\big)(2^{-\mu} m - y)}{\gamma!} \psi^\gamma(2^\mu x - m) \Bigg) dy \right\|_E .$$

The integrand is smooth, hence strongly measurable: g is an entire analytic function, and the terms in brackets are in $\mathcal{S}(\mathbb{R}^n)$. Note that the sum in γ is finite here. Then the Bochner Theorem allows us to push the E-norm inside the integral. We apply (4.16) and estimate

$$\int_{R^n} \|g(y)\|_E \bigg| \psi(2^\mu x - m)\rho(x-y) - \sum_{|\gamma|\le K} 2^{-\mu|\gamma|} \frac{(D^\gamma\rho)(2^{-\mu}m - y)}{\gamma!} \psi^\gamma(2^\mu x - m)\bigg|\, dy \le C c_K \mathbf{1}_{cQ_{\mu m}}(x) \int_{\mathbb{R}^n} \|g(y)\|_E \big(1 + |2^{-\mu}m - y|\big)^{-N} dy.$$

Recall that N can be made arbitrarily large. Moreover

$$1 + |x-y| \le \big(1 + |x - 2^{-\mu}m|\big)\big(1 + |2^{-\mu}m - y|\big) \le (1 + c2^{-\mu})\big(1 + |2^{-\mu}m - y|\big)$$

for all $x \in cQ_{\mu m}$. Using Theorem 3.5 we find some constant $C > 0$ s.t. for $w < p$

$$\begin{aligned} &\int_{\mathbb{R}^n} \|g(y)\|_E \big(1 + |2^{-\mu}m - y|\big)^{-N} dy \\ &\qquad \le C \sup_{y\in\mathbb{R}^n} \|g(y)\|_E \big(1 + |x-y|\big)^{-\frac{n}{w}} \cdot \int_{\mathbb{R}^n} \big(1 + |x-y|\big)^{-N+\frac{n}{w}} dy \\ &\qquad \le C\big(\mathbf{M}\|g\|_E^w\big)^{1/w}(x) \qquad \text{for all } x \in cQ_{\mu m}. \end{aligned} \tag{4.18}$$

The constant $C = C(\mu, n, p, w, K, N)$ does not depend on g. We obtain

$$\Big\| g(x) - \sum_{m\in\mathbb{Z}^n} \sum_{|\gamma|\le K} \lambda_m^\gamma \psi^\gamma(2^\mu x - m) \Big\|_E \le C c_K \big(\mathbf{M}\|g\|_E^w\big)^{1/w}(x) \cdot \sum_{m\in\mathbb{Z}^n} \mathbf{1}_{cQ_{\mu m}}(x)$$

for all $x \in \mathbb{R}^n$. Note that the m-sum on the RHS is uniformly bounded because only finitely many cubes $cQ_{\mu m}$ overlap. Now we take $L_q(\mathbb{R}^n)$-(quasi)norms on both sides. Since $w < p \le q$ (hence $q/w > 1$) we apply the Hardy-Littlewood maximal inequality (cf. Stein [20]) and obtain

$$\Big\| g(x) - \sum_{m\in\mathbb{Z}^n} \sum_{|\gamma|\le K} \lambda_m^\gamma \psi^\gamma(2^\mu x - m) \Big\|_{L_q(\mathbb{R}^n,E)} \le C c_K \big\| \mathbf{M}\|g\|_E^w \big\|_{L_{q/w}(\mathbb{R}^n)}^{1/w}$$
$$\text{and} \quad \big\| \mathbf{M}\|g\|_E^w \big\|_{L_{q/w}(\mathbb{R}^n)}^{1/w} \le C\|g\|_{L_q(\mathbb{R}^n,E)} \le C\|g\|_{L_p(\mathbb{R}^n,E)}.$$

In the last step we used the Nikol'skij inequality (3.4). Now c_K vanishes if $K \to \infty$. Therefore (4.11) converges strongly in $L_q(\mathbb{R}^n, E)$ for any $q \ge p$ as claimed.

To prove (4.12) we only need to modify this argument as little bit. Note first that the $\ell_p(\mathbb{Z}^n)$-(quasi)norm in m can also be realized like this

$$\begin{aligned} \sum_{m\in\mathbb{Z}^n} \|\lambda_m^\gamma\|_E^p &= \sum_{m\in\mathbb{Z}^n} \|\lambda_m^\gamma\|_E^p \cdot 2^{\mu n} \int_{\mathbb{R}^n} \mathbf{1}_{Q_{\mu m}}(x)\, dx \\ &= 2^{\mu n} \int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \|\lambda_m^\gamma\|_E^p \mathbf{1}_{Q_{\mu m}}(x)\, dx \end{aligned}$$

$$= 2^{\mu n} \int_{\mathbb{R}^n} \left(\sum_{m\in\mathbb{Z}^n} \|\lambda_m^\gamma\|_E \mathbf{1}_{Q_{\mu m}}(x) \right)^p dx$$

because the $Q_{\mu m}$ are pairwise disjoint. On the other hand, we find with (4.15)

$$2^{\mu|\gamma|} \|\lambda_m^\gamma\|_E \le C \int_{\mathbb{R}^n} \|g(y)\|_E \big(1 + |2^{-\mu}m - y|\big)^{-N} dy.$$

If we now continue with (4.18), we obtain (4.12). The proof is complete. □

Now we can present the subatomic decomposition for $B^s_{p,q}(\mathbb{R}^n, E)$

Theorem 4.3. *Let $0 < p, q \le \infty$ and $s > \sigma_p = \max\{n(1/p - 1), 0\}$. Then there exists a number $\kappa > 0$ s.t. for any $\mu \in \mathbb{N}$ with $\mu > \kappa$ any $f \in B^s_{p,q}(\mathbb{R}^n, E)$ can be decomposed into an infinite sum*

$$f(x) = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \sum_{\gamma\in\mathbb{N}_0^n} \lambda^\gamma_{\nu m} \mathbf{Q}^\gamma_{\nu m}(x) \qquad \text{in } \mathcal{S}'(\mathbb{R}^n, E). \tag{4.19}$$

The $\mathbf{Q}^\gamma_{\nu m}$ are $(s,p)_{-1} - \gamma$-quarks and the $\lambda^\gamma_{\nu m} \in E$ coefficients with

$$\sup_{\gamma\in\mathbb{N}_0^n} 2^{\mu|\gamma|} \left(\sum_{\nu=0}^{\infty} \left(\sum_{m\in\mathbb{Z}^n} \|\lambda^\gamma_{\nu m}\|_E^p \right)^{q/p} \right)^{1/q} < \infty. \tag{4.20}$$

Vice versa, if coefficients $\lambda^\gamma_{\nu m} \in E$ are given with (4.20), then the sum (4.19) converges in $\mathcal{S}'(\mathbb{R}^n, E)$ and defines an element in $B^s_{p,q}(\mathbb{R}^n, E)$. The inf in (4.20) over all admissible representations (4.19) is an equivalent (quasi)norm in $B^s_{p,q}(\mathbb{R}^n, E)$.

Remark 4.4. *This is Corollary 15.9 in Triebel [22].* One direction of the proof is straightforward: For $f \in \mathcal{S}'(\mathbb{R}^n, E)$ we apply Theorem 4.2 to the dyadic parts f_ν. Let us consider the family $g_\nu(y) = f_\nu(2^{-\nu}y)$ for $y \in \mathbb{R}^n$. From the definition of the Fourier transform we easily find that $\operatorname{supp} \hat{g}_\nu \subset B_2(0)$, $\forall \nu \in \mathbb{N}_0$. Hence the supports of $\hat{g}_\nu$ are all contained in one single fixed compact subset $K \subset \mathbb{R}^n$. Then there exists some $\kappa > 0$ s.t. for each $\mu > \kappa$ and all $\nu \in \mathbb{N}_0$ the entire analytic function g_ν can be decomposed into

$$g_\nu(y) = \sum_{m\in\mathbb{Z}^n} \sum_{\gamma\in\mathbb{N}_0^n} \tilde{\lambda}^\gamma_{\nu m} \psi^\gamma(2^\mu y - m),$$

with strong convergence in $L^q(\mathbb{R}^n, E)$ for all $q \ge p$, and coefficients $\tilde{\lambda}^\gamma_{\nu m} \in E$ with

$$\sup_{\gamma\in\mathbb{N}_0^n} 2^{\mu|\gamma|} \left(\sum_{m\in\mathbb{Z}^n} \|\tilde{\lambda}^\gamma_{\nu m}\|_E^p \right)^{1/p} \le C\|g_\nu\|_{L_p(\mathbb{R}^n,E)} = C2^{\nu\frac{n}{p}} \|f_\nu\|_{L_p(\mathbb{R}^n,E)}. \tag{4.21}$$

The constant C can be chosen independent of ν. Renormalizing we can write

$$f_\nu(x) = g_\nu(2^\nu x) = 2^{\mu(s-\frac{n}{p})} \sum_{m\in\mathbb{Z}^n} \sum_{\gamma\in\mathbb{N}_0^n} \underbrace{2^{\nu(s-\frac{n}{p})} \tilde{\lambda}^\gamma_{\nu m}}_{=\lambda^\gamma_{\nu m}} \underbrace{(\gamma q u)^{-1}_{\mu+\nu,m}(x)}_{=\mathbf{Q}^\gamma_{\nu m}(x)} \tag{4.22}$$

for $x \in \mathbb{R}^n$. This is (4.19). Now the proof of (4.20) is easy. From (4.21) we obtain

$$\begin{aligned}
&\sup_{\gamma\in\mathbb{N}_0^n} 2^{\mu|\gamma|}\left(\sum_{\nu=0}^{\infty}\left(\sum_{m\in\mathbb{Z}^n}\|\lambda_{\nu m}^{\gamma}\|_E^p\right)^{q/p}\right)^{1/q}\\
&\quad\le C\left(\sum_{\nu=0}^{\infty}\left(2^{\nu s}2^{-\nu\frac{n}{p}}\sup_{\gamma\in\mathbb{N}_0^n}2^{\mu|\gamma|}\left(\sum_{m\in\mathbb{Z}^n}\|\tilde\lambda_{\nu m}^{\gamma}\|_E^p\right)^{1/p}\right)^q\right)^{1/q}\\
&\quad\le C\left(\sum_{\nu=0}^{\infty}2^{\nu s q}\|f_\nu\|_{L_p(\mathbb{R}^n,E)}^q\right)^{1/q} = C\|f\|_{B^s_{p,q}(\mathbb{R}^n,E)}.
\end{aligned}$$

We used the Minkowski inequality. Proving the reverse direction is more elaborate, and we do not want to give the details here. We refer again to Triebel [22].

4.1.2. Proof of Theorem 2.2, Case I. Let us consider first the term $\bar f_{\mathbf{s},\nu}$ in (4.7). If we assume for the moment that $f_\nu(x,v)=\lambda(v)\mathbf{Q}(x)$ for suitable $\mathbf{Q}\in L_2(\mathbb{R}^n)$ and $\lambda\in E_1$, then $\hat f_\nu=\lambda\hat{\mathbf{Q}}$ is a measurable function, and we find

$$\langle \bar f_{\mathbf{s},\nu},\phi\rangle=\int_{\mathbb{R}^n}M(\xi)\hat{\mathbf{Q}}(\xi)\check\phi(\xi)\,d\xi \quad\text{with}\quad M(\xi)=\int_R\chi_{\mathbf{s}}(\xi,v)\big(\psi\lambda\big)(v)\,d\mu(v).$$

In that situation, we obtain $\bar f_{\mathbf{s},\nu}$ by simply applying the Fourier multiplier operator M to $\mathbf{Q}$. We can use Plancherel's Theorem to estimate

$$\big|\langle \bar f_{\mathbf{s},\nu},\phi\rangle\big|\le\|M\|_{L_\infty(\mathbb{R}^n)}\|\mathbf{Q}\|_{L_2(\mathbb{R}^n)}\|\phi\|_{L_2(\mathbb{R}^n)}.$$

By assumption, the product $\psi\lambda$ is bounded in $L_{\rho_1}(R,\mu)$. Therefore we may use the Hölder Inequality to estimate

$$\|M\|_{L_\infty(\mathbb{R}^n)}=\sup_{\xi\in\mathbb{R}^n}\left|\int_R\chi_{\mathbf{s}}(\xi,v)\big(\psi\lambda\big)(v)\,d\mu(v)\right|\le\|\psi\|_F\|\lambda\|_{E_1}\sup_{\xi\in\mathbb{R}^n}\|\mathbf{1}_{A_\delta(\xi)}\|_{L_{\rho_1'}(R,\mu)}$$

where $A_\delta(\xi)=\big\{v\in R\colon(\xi,v)\in A_\delta\big\}$. But the nondegeneracy condition (2.4) for the velocity field $\mathbf{a}$ bounds the measure of the set A_δ uniformly in $\xi\in\mathbb{R}^n$. Therefore

$$\|\bar f_{\mathbf{s},\nu}\|_{L_2(\mathbb{R}^n)}=\sup_{\phi\in\mathcal{S}(\mathbb{R}^n)}\|\phi\|_{L_2(\mathbb{R}^n)}^{-1}\big|\langle\bar f_{\mathbf{s},\nu},\phi\rangle\big|\le C\delta^{\alpha/\rho_1'}\|\psi\|_F\|\lambda\|_{E_1}\|\mathbf{Q}\|_{L_2(\mathbb{R}^n)}. \tag{4.23}$$

The conclusion is that $\bar f_{\mathbf{s},\nu}$ becomes small in $L_2(\mathbb{R}^n)$ for $\delta\to0$, if $\mathbf{Q}\in L_2(\mathbb{R}^n)$, $\lambda\in E_1$ and $\rho_1>1$. If $\rho_1=1$ the estimate does not depend on δ anymore.
We now consider the second term $\bar f_{\mathbf{r},\nu}$ in (4.8). If we assume again for the moment that $g_\nu(x,v)=\lambda(v)\mathbf{Q}(x)$ for suitable $\mathbf{Q}\in L_2(\mathbb{R}^n)$ and $\lambda\in E_2$, then $\hat g_\nu=\lambda\hat{\mathbf{Q}}$ is a measurable function, and we can write

$$\langle\bar f_{\mathbf{r},\nu},\phi\rangle=\int_{\operatorname{supp}\varphi_\nu}|\xi|^{-1}\bar M(\xi)\hat{\mathbf{Q}}(\xi)\check\phi(\xi)\,d\xi\quad\text{with }\bar M(\xi)=\int_R\bar\chi_{\mathbf{r}}(\xi,v)\big(\psi\lambda\big)(v)\,d\mu(v).$$

To estimate $\bar f_{\mathbf{r},\nu}$ in $L_2(\mathbb{R}^n)$, we need to find a L_∞-bound for the multiplier. We have $|\xi|^{-1}\le c2^{-\nu}$ in the domain of integration by the construction of φ_ν. Moreover

$$\|\bar M\|_{L_\infty(\mathbb{R}^n)}\le\|\psi\|_F\|\lambda\|_{E_2}\sup_{\xi\in\mathbb{R}^n}\left\|\big(i\mathbf{a}\cdot\xi/|\xi|\big)^{-1}\mathbf{1}_{R\setminus A_{\delta/2}(\xi)}\right\|_{L_{\rho_2'}(R,\mu)}.$$

Now we can use the following result.
LEMMA 4.5. *Assume (2.4). Then for every $\rho \geq 1$ with $\alpha < \rho$*

$$\sup_{\xi \in \mathbb{R}^n} \int_{R \setminus A_\delta(\xi)} \left| i\mathbf{a}(v) \cdot \xi / |\xi| \right|^{-\rho} d\mu(v) \leq C\delta^{\alpha - \rho}.$$

REMARK 4.6. *Here $C = C(\alpha, \rho)$ does not depend on δ. Estimates of this kind appear in many papers on velocity averaging, e.g. in [13], [16], [5].* We use Lemma 4.5 with $\rho = \rho_2'$ under the assumption that $\alpha/\rho_2' < 1$. By testing $\bar{f}_{\mathbf{r},\nu}$ against all functions $\phi \in \mathcal{S}(\mathbb{R}^n)$ we obtain

$$\|\bar{f}_{\mathbf{r},\nu}\|_{L_2(\mathbb{R}^n)} \leq C 2^{-\nu} \delta^{-1+\alpha/\rho_2'} \|\psi\|_F \|\lambda\|_{E_2} \|\mathbf{Q}\|_{L_2(\mathbb{R}^n)}.$$

We conclude that the dyadic elements $\bar{f}_{\mathbf{r},\nu}$ of the average vanish in $L_2(\mathbb{R}^n)$ like $2^{-\nu}$ if $\nu \to \infty$. This corresponds to a gain of regularity of one derivative (cf. Definition (3.7)). Note however that $\delta^{-1+\alpha/\rho_2'}$ becomes large as $\delta \to 0$.
Now we use Theorem 4.2, which tells that the dyadic blocks of f_ν and g_ν can be realized as tensor products. Recall (4.22) from §4.1.1: For suitable coefficients $\lambda^\gamma_{\nu m} \in E_1$ and $(s_1, p)_{-1} - \gamma$-quarks $\mathbf{Q}^\gamma_{\nu m}$ we have

$$f_\nu(x, v) = 2^{\mu(s - \frac{n}{p})} \sum_{m \in \mathbb{Z}^n} \sum_{\gamma \in \mathbb{N}_0^n} \lambda^\gamma_{\nu m}(v) \mathbf{Q}^\gamma_{\nu m}(x).$$

To control $\bar{f}_{\mathbf{s},\nu}$ in $L_2(\mathbb{R}^n)$, we can now use the triangle inequality and obtain

$$\|\bar{f}_{\mathbf{s},\nu}\|_{L_2(\mathbb{R}^n)} \leq C\delta^{\alpha/\rho_1'} \|\psi\|_F \sum_{m \in \mathbb{Z}^n} \sum_{\gamma \in \mathbb{N}_0^n} \|\lambda^\gamma_{\nu m}\|_{E_1} \|\mathbf{Q}^\gamma_{\nu m}\|_{L_2(\mathbb{R}^n)}. \tag{4.24}$$

Here the constant $C = C(\mu, n, p)$ does not depend on f or ν. Note now that quarks are normalized, i.e. there exists a constant $C = C(\mu)$ s.t.

$$\|\mathbf{Q}^\gamma_{\nu m}\|_{L_2(\mathbb{R}^n)} \leq C 2^{\sigma|\gamma|} 2^{-\nu s_1 + \nu n(\frac{1}{p} - \frac{1}{2})} \qquad \text{for some } \sigma > 0 \text{ and } \gamma \in \mathbb{N}_0^n, m \in \mathbb{Z}^n.$$

One nice feature of the subatomic decomposition in Theorem 4.2 is that we can choose the parameter $\mu \in \mathbb{N}$ as large as we want (at the expense of enlarging the constants, of course). For suitable μ and with $\kappa = n(\frac{1}{p} - \frac{1}{2})$ we therefore obtain the following estimate

$$\begin{aligned} \|\bar{f}_{\mathbf{s},\nu}\|_{L_2(\mathbb{R}^n)} &\leq C\delta^{\alpha/\rho_1'} 2^{-\nu s_1} 2^{\nu\kappa} \|\psi\|_F \\ &\quad \cdot \sup_{\gamma \in \mathbb{N}_0^n} 2^{\mu|\gamma|} \left(\sum_{m \in \mathbb{Z}^n} \|\lambda^\gamma_{\nu m}\|^p_{E_1} \right)^{1/p} \cdot \sum_{\gamma \in \mathbb{N}_0^n} 2^{-(\mu - \sigma)|\gamma|} \\ &\leq C\delta^{\alpha/\rho_1'} 2^{\nu\kappa} \|\psi\|_F \|f_\nu\|_{L_p(\mathbb{R}^n, E_1)}. \end{aligned} \tag{4.25}$$

Consult (4.21)/(4.22). Here we used the fact that $\ell_p(\mathbb{Z}^n) \hookrightarrow \ell_1(\mathbb{Z}^n)$ if $p \leq 1$. This is Case I of Theorem 2.2. A similar argument works for $\bar{f}_{\mathbf{r},\nu}$ with $\nu \geq 1$. We have

$$\|\bar{f}_{\mathbf{r},\nu}\|_{L_2(\mathbb{R}^n)} \leq C\delta^{-1+\alpha/\rho_2'} 2^{-\nu} 2^{\nu\kappa} \|\psi\|_F \|g_\nu\|_{L_p(\mathbb{R}^n, E_2)}.$$

The rest of the proof is only a matter of matching: For all $\nu \geq 1$ we want

$$\begin{aligned} 2^{\nu S} \|\bar{f}_{\mathbf{s},\nu}\|_{L_2(\mathbb{R}^n)} &\leq C\|\psi\|_F \cdot 2^{\nu s} \|f_\nu\|_{L_p(\mathbb{R}^n, E_1)} \quad \text{and} \\ 2^{\nu S} \|\bar{f}_{\mathbf{r},\nu}\|_{L_2(\mathbb{R}^n)} &\leq C\|\psi\|_F \cdot 2^{\nu(s - \tau)} \|g_\nu\|_{L_p(\mathbb{R}^n, E_2)}. \end{aligned} \tag{4.26}$$

To provide this, we choose the ansatz $\delta = 2^{-\nu\sigma}$ and solve the system of equations

$$S - \sigma\frac{\alpha}{\rho_1'} = s - \kappa$$
$$S - \sigma\left(-1 + \frac{\alpha}{\rho_2'}\right) = 1 + s - \tau - \kappa,$$

for (σ, S). We find

$$\sigma = (1-\tau)\left[1 + \alpha\Big(\frac{1}{\rho_2} - \frac{1}{\rho_1}\Big)\right]^{-1} \quad \text{and}$$
$$S = s - n\Big(\frac{1}{p} - \frac{1}{2}\Big) + (1-\tau)\frac{\alpha}{\rho_1'}\left[1 + \alpha\Big(\frac{1}{\rho_2} - \frac{1}{\rho_1}\Big)\right]^{-1}.$$

S is just the regularity of the average $\bar{f}$ we seek. Note also that $\sigma \geq 0$ and hence $\delta \leq 1$ because $\tau \leq 1$. Using (4.26) we obtain the estimate (4.1) for all $\nu \geq 1$.

4.2. Regularity - Case II. To prove estimate (4.1) for $1 < p < \frac{n}{n-1}$ we use the fact that the action of some homogeneous Fourier multiplier operator on smooth functions can be rewritten in terms of the well known Radon transform.

4.2.1. Some remarks on the Radon transform. The Radon transform $\mathbf{R}$ maps a function $\Phi \in \mathcal{S}(\mathbb{R}^n)$ to the average of Φ over all $n-1$-dimensional hyperplanes in $\mathbb{R}^n$. Every such hyperplane is characterized by (1) the unit normal vector $\omega \in S^{n-1}$ and (2) the distance $r \geq 0$ between hyperplane and origin. We therefore define

$$\mathbf{R}\Phi(\omega, r) = \int_{\omega\cdot x = r} \Phi(x)\, dS(x) \qquad \text{for } (\omega, r) \in S^{n-1} \times [0, \infty). \tag{4.27}$$

Here dS is the induced Lebesgue measure. As a synonym we will also write $\tilde{\Phi} = \mathbf{R}\Phi$. Putting $\tilde{\Phi}(\omega, r) = \tilde{\Phi}(-\omega, -r)$, the Radon transform can be extended to a function on $S^{n-1} \times \mathbb{R}$. We have the following relationship with the Fourier transform

$$\begin{aligned}(2\pi)^n \check{\Phi}(s\omega) &= \int_{\mathbb{R}^n} e^{is\omega\cdot x}\Phi(x)\, dx = \int_{\mathbb{R}} e^{isr}\left(\int_{\omega\cdot x = r}\Phi(x)\, dS(x)\right) dr \\ &= \int_{\mathbb{R}} e^{isr}\tilde{\Phi}(\omega, r)\, dr = 2\pi\mathbf{F}^{-1}\tilde{\Phi}(\omega, s) \qquad \text{for } (\omega, s) \in S^{n-1} \times \mathbb{R}.\end{aligned} \tag{4.28}$$

Let $m \in L_\infty(\mathbb{R}^n)$ be an even homogeneous function of degree zero. For simplicity we assume $m \in C^\infty(S^{n-1})$. Choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\check{\varphi} \subset \mathbb{R}^n \backslash B_1(0)$. Then

$$\begin{aligned}\mathbf{F}[m\check{\varphi}](x) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} m(\xi)\check{\varphi}(\xi)\, d\xi \\ &= \int_{S^{n-1}} m(\omega)\int_0^\infty e^{-ix\cdot s\omega} s^{n-1}\check{\varphi}(s\omega)\, ds\, d\omega.\end{aligned} \tag{4.29}$$

Note that $m\check{\varphi} \in \mathcal{S}(\mathbb{R}^n)$. Now we define $\Phi \in \mathcal{S}(\mathbb{R}^n)$ by $\check{\Phi}(\xi) = |\xi|^{n-1}\check{\varphi}(\xi)$, $\forall \xi \in \mathbb{R}^n$. Since m is even, we obtain after a substitution $t = -s$ and $\sigma = -\omega$

$$\int_{S^{n-1}} m(\omega)\int_0^\infty e^{-ix\cdot s\omega}\check{\Phi}(s\omega)\, ds\, d\omega = \int_{S^{n-1}} m(\sigma)\int_{-\infty}^0 e^{-ix\cdot t\sigma}\check{\Phi}(t\sigma)\, dt\, d\sigma$$

We may therefore extend the s-integral in (4.29) to the whole real line, if we allow for an extra factor 1/2. If we now use equality (4.28), we find

$$
\begin{aligned}
\mathbf{F}[m\check{\varphi}](x) &= \frac{1}{2(2\pi)^{n-1}} \int_{S^{n-1}} m(\omega) \int_{\mathbb{R}} e^{-isx\cdot\omega} \mathbf{F}^{-1}\tilde{\Phi}(\omega, s)\, ds\, d\omega \\
&= \frac{1}{2(2\pi)^{n-1}} \int_{S^{n-1}} m(\omega)\tilde{\Phi}(\omega, x\cdot\omega)\, d\omega. \qquad (4.30)
\end{aligned}
$$

The RHS is simply the average of $\tilde{\Phi}$ over all hyperplanes containing a given point $x \in \mathbb{R}^n$. Now $\mathbf{F}[|\cdot|^{n-1}\check{\varphi}]$ is just a power of the Laplacian, $\Delta^{\frac{n-1}{2}}\varphi$, times some constant (cf. Stein [20]). Hence we obtain for $m(\xi) = 1$ the following identity

$$
\varphi(x) = c\Delta^{\frac{n-1}{2}} \int_{S^{n-1}} \tilde{\varphi}(\omega, x\cdot\omega)\, d\omega.
$$

This gives an inversion formula for the Radon transform.

4.2.2. Proof of Theorem 2.2, Case II. Consider first $\hat{f}_{\mathbf{s},\nu}$ in (4.7). Following the proof of Theorem 3.1 we obtain for $\chi_{\mathbf{s}}\hat{f}_\nu \in \mathcal{S}'(\mathbb{R}^n, E)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$
\langle \chi_{\mathbf{s}}\hat{f}_\nu, \check{\phi}\rangle = \lim_{\epsilon\to 0} \langle \chi_{\mathbf{s}}\hat{f}_{\nu,\epsilon}, \check{\phi}\rangle \quad \text{with } \hat{f}_{\nu,\epsilon} = \hat{f}_\nu \star \eta_\epsilon,
$$

with η_ϵ as in §3.1. Since $\hat{f}_\nu = \varphi_\nu \hat{f}$ is compactly supported, $\hat{f}_{\nu,\epsilon} \in \mathcal{S}(\mathbb{R}^n, E)$. We choose now some $\rho_1 \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp}\rho_1 \subset B_5(0)\backslash B_{1/5}(0)$ and $\rho_1(\xi) = 1$ for $\xi \in \operatorname{supp}\varphi_1$, and put $\rho_\nu(\xi) = \rho_1(2^{-\nu+1}\xi)$ for $\nu \geq 1$. Then we have for all $\epsilon > 0$

$$
\langle \chi_{\mathbf{s}}\hat{f}_{\nu,\epsilon}, \check{\phi}\rangle = \int_{\mathbb{R}^n} \big(\chi_{\mathbf{s}}\rho_\nu\check{\phi}\big)(\xi)\cdot \hat{f}_{\nu,\epsilon}(\xi)\, d\xi = \int_{\mathbb{R}^n} \mathbf{F}[\chi_{\mathbf{s}}\rho_\nu\check{\phi}](x)\cdot\check{\eta}(\epsilon x) f_\nu(x)\, dx
$$

in E. Note here that $f_\nu \in \mathcal{O}_M(\mathbb{R}^n, E)$ and $\mathbf{F}[\chi_{\mathbf{s}}\rho_\nu\check{\phi}] \in \mathcal{S}(\mathbb{R}^n, L_\infty(R,\mu))$. Using the dominated convergence theorem we may let $\epsilon \to 0$ to find

$$
\langle \chi_{\mathbf{s}}\hat{f}_\nu, \check{\phi}\rangle = \int_{\mathbb{R}^n} \mathbf{F}[\chi_{\mathbf{s}}\rho_\nu\check{\phi}](x)\cdot f_\nu(x)\, dx, \quad \forall\phi\in\mathcal{S}(\mathbb{R}^n).
$$

Now let us consider averages again. Since $\chi_{\mathbf{s}}(\xi, v)$ for fixed $v \in R$ is even and homogeneous in ξ, we can use (4.30). We obtain with $\Phi_\nu = \mathbf{F}[|\cdot|^{n-1}\rho_\nu\check{\phi}]$, $\nu \geq 1$

$$
\langle \bar{f}_{\mathbf{s},\nu}, \phi\rangle = \frac{1}{2(2\pi)^{n-1}} \int_R \int_{\mathbb{R}^n} \left(\int_{S^{n-1}} \chi_{\mathbf{s}}(\omega, v)\tilde{\Phi}_\nu(\omega, x\cdot\omega)\, d\omega \right) f_\nu(x,v)\psi(v)\, dx\, d\mu(v). \qquad (4.31)
$$

We must show that the integral exists (absolute integrability). We first do the integration in v. Then the Hölder inequality and our assumptions give us an estimate

$$
\int_R |\psi(v) f_\nu(x,v)\chi_{\mathbf{s}}(\omega, v)|\, d\mu(v) \leq \|\psi\|_F \|f_\nu(x)\|_{E_1} \|\chi_{\mathbf{s}}(\omega)\|_{L_{\rho_1'}(R,\mu)} \qquad (4.32)
$$

for $(x,\omega) \in \mathbb{R}^n \times S^{n-1}$. Since $\chi_{\mathbf{s}}(\omega, v) \leq \mathbf{1}_{A_\delta(\omega)}(v)$ for all $v \in R$, where

$$
A_\delta(\omega) = \big\{ v \in R \colon |\mathbf{a}(v)\cdot\omega| \leq \delta \big\} \quad \text{with } \omega \in S^{n-1},
$$

assumption 2.4 on the nondegeneracy of the velocity field yields a bound $C\delta^{\alpha/\rho_1'}$ uniformly in ω for the last term on the RHS of (4.32). Now we integrate over the sphere

S^{n-1}. Recall that the Radon transform is defined as an integral over hyperplanes. Therefore we may write

$$\big|\mathbf{R}\Phi_\nu(\omega,r)\big| = \left|\int_{\omega\cdot x=r}\Phi_\nu(x)\,dS(x)\right| \le \int_{\omega\cdot x=r}|\Phi_\nu(x)|\,dS(x) = \mathbf{R}|\Phi_\nu|(\omega,r)$$

for $(\omega,r)\in S^{n-1}\times\mathbb{R}$. I.e. the Radon transform of some function can be estimated in absolute value by the Radon transform of the absolute value of that function. Now we use the following result (cf. Ramm & Katsevich [19], Lemma 2.1.1.)

$$\int_{S^{n-1}}\mathbf{R}|\Phi_\nu|(\omega,x\cdot\omega)\,d\omega = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_{\mathbb{R}^n}\frac{|\Phi_\nu(y)|}{|x-y|}\,dy$$

for all $x\in\mathbb{R}^n$. The operator on the RHS is a fractional integration of order $n-1$. We refer to Stein [20], Kapitel VIII/4.2. Finally, we do the integration in x and use the Hölder inequality again. Then we can estimate

$$\big|\big\langle\bar f_{\mathbf{s},\nu},\phi\big\rangle\big| \le C\delta^{\alpha/\rho_1'}\|\psi\|_F\|f_\nu\|_{L_p(\mathbb{R}^n,E_1)}\left\|\int_{\mathbb{R}^n}\frac{|\Phi_\nu(y)|}{|\cdot-y|}\,dy\right\|_{L_{p'}(\mathbb{R}^n)}.$$

Let $1<P<\infty$ be given with $\frac{1}{P}=\frac{1}{p}-\frac{n-1}{n}$. Then (cf. Stein [20])

$$\left\|\int_{\mathbb{R}^n}\frac{|\Phi_\nu(y)|}{|\cdot-y|}\,dy\right\|_{L_{p'}(\mathbb{R}^n)} \le C\|\Phi_\nu\|_{L_{P'}(\mathbb{R}^n)}.$$

The constant does not depend on Φ_ν. Since $|\cdot|^{n-1}\rho_\nu = 2^{(\nu-1)(n-1)}[|\cdot|^{n-1}\rho_1](2^{-\nu+1}\cdot)$

$$\|\Phi_\nu\|_{L_{P'}(\mathbb{R}^n)} \le C2^{(\nu-1)(n-1)}\|\phi\|_{L_{P'}(\mathbb{R}^n)}$$

for $\nu\ge 1$ with $C=\|\mathbf{F}[|\cdot|^{n-1}\rho_1]\|_{L_1(\mathbb{R}^n)}$ (the Young inequality). This proves absolute integrability for (4.31). We test $\bar f_{\mathbf{s},\nu}$ against all $\phi\in\mathcal{S}(\mathbb{R}^n)$ and use the fact that the Schwartz class is dense in $L_{P'}(\mathbb{R}^n)$. We obtain the following estimate

$$\begin{aligned}\|\bar f_{\mathbf{s},\nu}\|_{L_P(\mathbb{R}^n)} &= \sup_{\phi\in\mathcal{S}(\mathbb{R}^n)}\|\phi\|^{-1}_{L_{P'}(\mathbb{R}^n)}\Big|\big\langle\bar f_{\mathbf{s},\nu},\phi\big\rangle\Big| \\ &\le C2^{\nu(n-1)}\delta^{\alpha/\rho_1'}\|\psi\|_F\|f_\nu\|_{L_p(\mathbb{R}^n,E_1)}.\end{aligned} \tag{4.33}$$

Of course, we now want to apply a similar argument to g_ν. As above, we obtain

$$\big\langle\chi_{\mathbf{r}}\hat g_\nu/(i\mathbf{a}\cdot\xi),\check\phi\big\rangle = 2^{-\nu+1}\int_{\mathbb{R}^n}\mathbf{F}[\bar\chi_{\mathbf{r}}\bar\rho_\nu\check\phi](x)\cdot g_\nu(x)\,dx,\quad \forall\varphi\in\mathcal{S}(\mathbb{R}^n).$$

Here, $\bar\rho_1(\xi)=\rho_1(\xi)/|\xi|$ and $\bar\rho_\nu(\xi)=\bar\rho_1(2^{-\nu+1}\xi)$ for all $\xi\in\mathbb{R}^n$ and $\nu\in\mathbb{N}$. But now we have a problem: for v fixed the function $\bar\chi_{\mathbf{r}}$ is homogeneous, but not even in ξ. To circumvent this difficulty we use the Riesz transforms $\mathcal{R}_j$ with $j=1\ldots n$. These mappings are defined e.g. for all $\phi\in\mathcal{S}(\mathbb{R}^n)$ s.t. $\operatorname{supp}\check\phi\subset\mathbb{R}^n\backslash B_1(0)$ by

$$\mathbf{F}^{-1}[\mathcal{R}_j\phi](\xi) = i\xi_j|\xi|^{-1}\check\phi(\xi),\quad\forall\xi\in\mathbb{R}^n.$$

Applying these operators twice gives the following identity $\phi=-\sum_{j=1}^n\mathcal{R}_j^2\phi$. The $\mathcal{R}_j$ are $L_p(\mathbb{R}^n)$-continuous for $1<p<\infty$. We refer again to Stein [20] for more information and proofs. Let us define now $\Phi_{\nu,j}\in\mathcal{S}(\mathbb{R}^n)$ and $\bar\chi_{\mathbf{r},j}$ as follows

$$\check\Phi_{\nu,j}(\xi) = i\xi_j|\xi|^{n-2}\bar\rho_\nu(\xi)\check\phi(\xi)\qquad\text{and}\qquad\bar\chi_{\mathbf{r},j}(\xi,v) = i\xi_j|\xi|^{-1}\bar\chi_{\mathbf{r}}(\xi,v)$$

for all $(\xi, v) \in \mathbb{R}^n \times R, j = 1 \ldots n$ and $\nu \geq 1$. Then we obtain

$$\langle \bar{f}_{\mathbf{r},\nu}, \phi \rangle = C \sum_{j=1}^{n} \int_R \int_{\mathbb{R}^n} \left(\int_{S^{n-1}} \bar{\chi}_{\mathbf{r},j}(\omega, v) \tilde{\Phi}_{\nu,j}(\omega, x \cdot \omega) \, d\omega \right) g_\nu(x, v) \psi(v) \, dx \, d\mu(v).$$

Note that $\bar{\chi}_{\mathbf{r},j}$ for $v \in R$ fixed is an even function in ξ. Again, we do the integration in v first and obtain for all $(x, \omega) \in \mathbb{R}^n \times S^{n-1}$ the following estimate

$$\int_R |\psi(v) g_\nu(x, v) \bar{\chi}_{\mathbf{r}}(\omega, v)| \, d\mu(v) \leq \|\psi\|_F \|g_\nu(x)\|_{E_2} \left(\int_R \frac{\mathbf{1}_{R \setminus A_{\delta/2}(\omega)}(v)}{|i\mathbf{a}(v) \cdot \omega|^{\rho_2'}} d\mu(v) \right)^{1/\rho_2'} .$$

Using the nondegeneracy of the velocity field and Lemma 4.5 again, it is easy to show that the last factor is bounded by $C\delta^{-1+\alpha/\rho_2'}$ uniformly in ω. Proceeding now as we did above, we find a constant C s.t.

$$\|\bar{f}_{\mathbf{r},\nu}\|_{L_P(\mathbb{R}^n)} \leq C 2^{\nu(n-2)} \delta^{-1+\alpha/\rho_2'} \|\psi\|_F \|g_\nu\|_{L_p(\mathbb{R}^n, E_2)} \tag{4.34}$$

for $\nu \geq 1$. We used $\|\mathcal{R}_j \phi\|_{L_{P'}(\mathbb{R}^n)} \leq C \|\phi\|_{L_{P'}(\mathbb{R}^n)}$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ and j.

5. Proofs - Compactness.

First we must introduce some terminology.

5.1. Rearrangement-invariant Banach Function Spaces.

The usual Lebesgue spaces $L_p(R, \mu)$ over the finite measure space (R, μ) are just instances of more general so-called rearrangement-invariant Banach function spaces. We collect here some basic facts about these spaces, but refer to Bennett & Sharpley [4] for more details.

Consider the vector space $\mathcal{M}(R, \mu)$ of μ-measurable mappings from R into $\mathbb{R}$ (or $\mathbb{C}$), where as usual functions which coincide μ-almost everywhere (μ-a.e.) are identified. Denote by $\mathcal{M}^+(R, \mu)$ the set of all $f \in \mathcal{M}(R, \mu)$ with $f \geq 0$, and for some μ-measurable subset $A \subset R$ let $\mathbf{1}_A$ be its characteristic function.

Definition 5.1. *A mapping $\rho: \mathcal{M}^+(R, \mu) \to [0, \infty]$ is called a Banach function norm, if for all $f, f_n \in \mathcal{M}^+(R, \mu)$, $a \geq 0$ and μ-measurable sets $A \subset R$ we have:*

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ *μ-a.e.,*
$\rho(af) = a\rho(f)$ *and*
$\rho(f_1 + f_2) \leq \rho(f_1) + \rho(f_2)$
(P2) $0 \leq f_1 \leq f_2$ *μ-a.e.* $\Rightarrow \rho(f_1) \leq \rho(f_2)$
(P3) $0 \leq f_n \nearrow f_0$ *μ-a.e.* $\Rightarrow \rho(f_n) \nearrow \rho(f_0)$
(P4) $\mu(A) < \infty \Rightarrow \rho(\mathbf{1}_A) < \infty$
(P5) $\mu(A) < \infty \Rightarrow \int_A f \, d\mu < C\rho(f)$

Here $C = C(\rho, A)$ is a constant not depending on f.

Definition 5.2. *Let ρ be a Banach function norm. Then the Banach function space $E = E(\rho)$ is defined as the space of all $f \in \mathcal{M}(R, \mu)$ s.t. $\|f\|_E = \rho(|f|) < \infty$.*

For each Banach function norm ρ we can define an associated norm ρ' via

$$\rho'(g) = \sup \left\{ \int_R fg \, d\mu : f \in \mathcal{M}^+(R, \mu), \rho(f) \leq 1 \right\},$$

for all $g \in \mathcal{M}^+(R, \mu)$. The associated norm ρ' also has all the properties (P1)–(P5) in definition 5.1 and therefore generates a Banach function space $E' = E(\rho')$ assciated to E. The generalized Hölder inequality holds:

Theorem 5.3. *Let E be a Banach function space and E' the space associated to E. If $f \in E$ and $g \in E'$, then the product fg is absolutely integrable, and we have*

$$\int_R |fg|\, d\mu \le \|f\|_E \|g\|_{E'}. \tag{5.1}$$

Definition 5.4. *The distribution function μ_f of $f \in \mathcal{M}(R,\mu)$ is defined by*

$$\mu_f(\lambda) = \mu\left\{x \in R\colon |f(x)| > \lambda\right\}, \quad \forall \lambda \ge 0.$$

Note that μ_f depends only on $|f|$. If a second finite measure space (S,ν) is given, then two functions $f \in \mathcal{M}(R,\mu)$ and $g \in \mathcal{M}(S,\nu)$ are called equimeasurable, if their distribution functions are identical, i.e. if $\mu_f(\lambda) = \nu_g(\lambda)$ for all $\lambda \ge 0$.

Definition 5.5. *For $f \in \mathcal{M}(R,\mu)$ let f^* be the function on $[0,\infty)$ defined by*

$$f^*(t) = \inf\left\{\lambda\colon \mu_f(\lambda) \le t\right\}, \quad \forall t \ge 0.$$

f^* is called the decreasing rearrangement of f.

In other words, f^* is a decreasing function on $[0,\infty)$ with the same distribution function as f itself. Here we use the convention $\inf\emptyset = \infty$, i.e. if $\mu_f(\lambda) > t$ for all $\lambda \ge 0$, then $f^*(t) = \infty$. For a finite measure space (R,μ) the distribution function μ_f is always bounded. Then f^* is a function on the interval $[0,\mu(R)]$.

Definition 5.6. *Let ρ be a Banach function norm over some finite measure space (R,μ). Then ρ is called rearrangement-invariant, if $\rho(f) = \rho(g)$ for all pairs of equimeasurable functions $f,g \in \mathcal{M}(R,\mu)$. In that case we also call the Banach function space $E = E(\rho)$ defined by ρ rearrangement-invariant.*

Let us now assume that the finite measure space (R,μ) is also nonatomic, i.e. no single point carries a positive measure. This excludes Dirac measures.

Definition 5.7. *Let E be a rearrangement-invariant Banach function space over some finite nonatomic measure space (R,μ). For every t inside the intervall $[0,\mu(R)]$ let A be a μ-measurable subset of R with $\mu(A) = t$. Then the function*

$$\varphi_E\colon t \mapsto \|\mathbf{1}_A\|_E \tag{5.2}$$

is called the fundamental function of E.

Since for every $B \subset R$ with $\mu(B) = t$ the functions $\mathbf{1}_A$ and $\mathbf{1}_B$ are equimeasurable, and since E is assumed rearrangement-invariant, φ_E is well defined.

The fundamental function φ_E of a rearrangement-invariant Banach function space E is quasiconcave, i.e. φ_E is increasing, $\varphi_E(t) = 0$ iff $t = 0$, and $\varphi_E(t)/t$ is decreasing. From the quasiconcavity of the function continuity in $(0,\mu(R)]$ follows. Nevertheless, a discontinuity at zero is still possible.

Theorem 5.8. *Let E be a rearrangement-invariant Banach function space over the finite nonatomic measure space (R,μ) and E' the associated space. Then*

$$\varphi_E(t)\varphi_{E'}(t) = t, \quad \forall t \in [0,\mu(R)].$$

We refer to Section II.5 in [4]. Let us discuss two examples. The fundamental function of the Lebesgue space $L^p(R,\mu)$ with $1 \le p < \infty$ is given by $\varphi_{L^p}(t) = t^{1/p}$ for $t \in [0,\mu(R)]$. However if $p = \infty$, then

$$\varphi_{L^\infty}(t) = \begin{cases} 0 & \text{for } t = 0 \\ 1 & \text{for } t \in (0,\mu(R)], \end{cases}$$

since the characteristic function of a null set is equivalent to the zero function. In that case the fundamental function is discontinuous at zero.
In applications the Orlicz space $L\log L(R,\mu)$ plays an important role. That is the rearrangement-invariant Banach function space defined by the norm

$$\|f\|_{L\log L(R,\mu)} = -\int_0^{\mu(R)} f^*(t)\log\big(t/\mu(R)\big)\,dt.$$

For the fundamental function we have

$$\varphi_{L\log L}(t) = t\Big(1-\log\big(t/\mu(R)\big)\Big), \quad \forall t\in[0,\mu(R)].$$

The space associated to $L\log L(R,\mu)$ is the space $\exp L(R,\mu)$ of exponentially integrable functions. The corresponding fundamental function is given by

$$\varphi_{\exp L}(t) = \frac{1}{1-\log\big(t/\mu(R)\big)}, \quad \forall t\in[0,\mu(R)].$$

Note that $\varphi_{\exp L}$ is continuous at zero (but not differentiable). That is remarkable because for all $1<p<\infty$ the following inclusions hold (since $\mu(R)<\infty$)

$$L^\infty(R,\mu)\hookrightarrow \exp L(R,\mu)\hookrightarrow L^p(R,\mu)\hookrightarrow L\log L(R,\mu)\hookrightarrow L^1(R,\mu).$$

Although $\exp L$ and L^∞ are so close that no other L^p-space fits between them, there is a considerable difference in their respective fundamental functions.

5.2. Proof of Theorem 2.5. Again, we briefly recall our assumptions. Let E_1 and E_2 be two rearrangement-invariant Banach function spaces over some nonatomic finite measure spaces. Assume we are given a sequence of pairs

$$f^{(k)}\in B^s_{p,q}(\mathbb{R}^n,E_1) \quad\text{and}\quad g^{(k)}\in B^{s-\tau}_{p,q}(\mathbb{R}^n,E_2)$$

satisfying the transport equation (2.1) in $\mathcal{S}'(\mathbb{R}^n,E)$ for $E=E_1+E_2$. We fix a weight ψ in some subset $F\subset E'$ s.t. multiplication with ψ maps E_1 continuously into some rearrangement-invariant Banach function space G_1, and similarly E_2 into some G_2. If now the velocity field satisfies a nondegeneracy condition, and if the fundamental function of G_1' is continuous at zero, we will show that the sequence of averages $\bar f^{(k)}$ is precompact in a suitable local Besov space $B^{S,loc}_{P,q}(\mathbb{R}^n)$.
To simplify notation, we will drop the index k in the following. We already know that there exists a decomposition of the average $\bar f$ into $\bar f=\sum_{\nu=0}^\infty \bar f_\nu$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\bar f_\nu=\bar f_{\mathbf{s},\nu}+\bar f_{\mathbf{r},\nu}$ for $\nu\ge 1$. In contrast to our approach in §4 we will now choose the splitting parameter δ independent of ν. Then we can write $\bar f=\bar f_0+\bar F_{\mathbf{s}}+\bar F_{\mathbf{r}}$ with $\bar F_{\mathbf{s}}=\sum_{\nu=1}^\infty \bar f_{\mathbf{s},\nu}$ and $\bar F_{\mathbf{r}}=\sum_{\nu=1}^\infty \bar f_{\mathbf{r},\nu}$. We claim that $\bar f_0\in B^\sigma_{P,q}(\mathbb{R}^n)$ for arbitrary $\sigma\in\mathbb{R}$ and P as given above. Moreover

$$\bar F_{\mathbf{s}}\in B^S_{P,q}(\mathbb{R}^n) \quad\text{and}\quad \bar F_{\mathbf{r}}\in B^{S+\epsilon}_{P,q}(\mathbb{R}^n)$$

with the S from above and $\epsilon=1-\tau>0$. To see that, let us first consider the dyadic components $\bar F_{\mathbf{s},\nu}$. As an immediate consequence of the support properties of the family φ_ν with $\nu\ge 0$ we realize that

$$\bar F_{\mathbf{s},\nu}=\mathbf{F}^{-1}[\varphi_\nu\mathbf{F}\bar F_{\mathbf{s}}]=\begin{cases}\mathbf{F}^{-1}[\varphi_0\mathbf{F}\bar f_{\mathbf{s},1}] & \text{if } \nu=0\\ \mathbf{F}^{-1}[\varphi_1\mathbf{F}(\bar f_{\mathbf{s},1}+\bar f_{\mathbf{s},2})] & \text{if } \nu=1\\ \bar f_{\mathbf{s},\nu} & \text{for all } \nu\ge 2.\end{cases}$$

Since $P \geq 1$ we can now apply the Young inequality to obtain the estimates

$$\begin{aligned}&\|\bar{F}_{\mathbf{s},0}\|_{L_P(\mathbb{R}^n)} \leq C\|\bar{f}_{\mathbf{s},1}\|_{L_P(\mathbb{R}^n)},\\ &\|\bar{F}_{\mathbf{s},1}\|_{L_P(\mathbb{R}^n)} \leq C\big\{\|\bar{f}_{\mathbf{s},1}\|_{L_P(\mathbb{R}^n)} + \|\bar{f}_{\mathbf{s},2}\|_{L_P(\mathbb{R}^n)}\big\} \quad \text{and}\\ &\|\bar{F}_{\mathbf{s},\nu}\|_{L_P(\mathbb{R}^n)} = \|\bar{f}_{\mathbf{s},\nu}\|_{L_P(\mathbb{R}^n)} \quad \text{for all } \nu \geq 2.\end{aligned}$$

But the L_P-norm of $\bar{f}_{\mathbf{s},\nu}$ has already been estimated in the last sections. Using the generalized Hölder inequality for the Banach function space G_1 we obtain the following analogue of the estimates (4.25) and (4.33)

$$\|\bar{f}_{\mathbf{s},\nu}\|_{L_P(\mathbb{R}^n)} \leq C2^{\nu(\frac{1}{p}-\frac{1}{P})} \sup_{\xi\in\mathbb{R}^n} \|\mathbf{1}_{A_\delta(\xi)}\|_{G_1'}\|\psi\|_F\|f_\nu\|_{L_p(\mathbb{R}^n,E_1)}.$$

The constant does not depend on f or ν. But the G_1'-norm of the characteristic function of some set A with measure $s \geq 0$ is just the fundamental function $\varphi_{G_1'}(s)$. And since this function is increasing, we can use the nondegeneracy condition for $\mathbf{a}$ to obtain the following estimate (cf. (2.9))

$$\|\bar{F}_{\mathbf{s}}\|_{B^S_{P,q}(\mathbb{R}^n)} \leq C\varphi_{G_1'}\big(\eta(\delta)\big)\|\psi\|_F\|f\|_{B^s_{p,q}(\mathbb{R}^n,E_1)}. \tag{5.3}$$

Again the constant C does not depend on f or δ. In the same way, we can find a bound for the second term $\bar{F}_{\mathbf{r}}$. We use the generalized Hölder inequality for G_2, estimate $|i\mathbf{a}\cdot\omega|^{-1}\mathbf{1}_{R\setminus A_{\delta/2}(\omega)}$ by $C\delta^{-1}$, and obtain

$$\|\bar{F}_{\mathbf{r}}\|_{B^{S+\epsilon}_{P,q}(\mathbb{R}^n)} \leq C\delta^{-1}\|\psi\|_F\|g\|_{B^{s-\tau}_{p,q}(\mathbb{R}^n,E_2)}. \tag{5.4}$$

Finally, note that most dyadic parts of $\bar{f}_0$ vanish by the construction of φ_ν. Using the Young, the Nikol'skij and the Hölder inequality (3.4)/(5.1) we then obtain

$$\|\bar{f}_0\|_{B^\sigma_{P,q}(\mathbb{R}^n)} \leq C\|\psi\|_F\|f\|_{B^s_{p,q}(\mathbb{R}^n,E_1)}, \quad \forall\sigma\in\mathbb{R}.$$

Let us now consider sequences $f^{(k)}, g^{(k)}$ with (2.8), satisfying a transport equation (2.1). Let $\chi \in \mathcal{D}(\mathbb{R}^n)$ be some test function with compact support. Then we may decompose $\chi\bar{f}^{(k)}$ into three parts $\chi\bar{F}^{(k)}_{\mathbf{s}}, \chi\bar{F}^{(k)}_{\mathbf{r}}$ and $\chi\bar{f}^{(k)}_0$, and estimate as follows:

I. By assumption, the fundamental function of the rearrangement-invariant Banach function space G_1' is continuous at zero, and $\lim_{\delta\to 0}\eta(\delta) = 0$. Therefore

$$\|\chi\bar{F}^{(k)}_{\mathbf{s}}\|_{B^S_{P,q}(\mathbb{R}^n)} \leq C\|\bar{F}^{(k)}_{\mathbf{s}}\|_{B^S_{P,q}(\mathbb{R}^n)} \leq C\varphi_{G_1'}\big(\eta(\delta)\big)\|\psi\|_F\|f^{(k)}\|_{B^s_{p,q}(\mathbb{R}^n,E_1)} \longrightarrow 0$$

uniformly in k, if $\delta \to 0$: The sequence $f^{(k)}$ is uniformly bounded and the constants are independent of k and δ. We also used Theorem 3.12 and (5.3). We conclude that the first part of $\bar{f}^{(k)}$ becomes small choosing the parameter δ suitably.

II. Because of Theorem 3.12 and (5.4) there exists a number $C > 0$ s.t.

$$\|\chi\bar{F}^{(k)}_{\mathbf{r}}\|_{B^{S+\epsilon}_{P,q}(\mathbb{R}^n)} \leq C\|\bar{F}^{(k)}_{\mathbf{r}}\|_{B^{S+\epsilon}_{P,q}(\mathbb{R}^n)} \leq C\delta^{-1}\|\psi\|_F\|g^{(k)}\|_{B^{s-\tau}_{p,q}(\mathbb{R}^n,E_2)} \tag{5.5}$$

with C independent of δ, k and $g^{(k)}$. Therefore, for every fixed $\delta > 0$ the sequence $\chi\bar{F}^{(k)}_{\mathbf{r}}$ is uniformly bounded in $B^{S+\epsilon}_{P,q}(\mathbb{R}^n)$ for some $\epsilon > 0$. And since the supports of all functions of the sequence are contained in one single compact subset of $\mathbb{R}^n$, we can use e.g. Theorem 3.3.2/1 in Edmunds & Triebel [11] to conclude that $\chi\bar{F}^{(k)}_{\mathbf{r}}$ is precompactly contained in $B^S_{P,q}(\mathbb{R}^n)$. Similarly we proceed for $\chi\bar{f}^{(k)}_0$.

Therefore, the sequence of averages $\chi\bar{f}^{(k)}$ can be decomposed into three parts, two of which are precompact in $B^S_{P,q}(\mathbb{R}^n)$, while the third goes to zero uniformly w.r.t. k, as $\delta \to 0$. Hence $\chi\bar{f}^{(k)}$ itself is precompact as claimed. The proof is complete.

REMARK 5.9. *Note that the fine structure of G_2 plays no role in the proof of Theorem 2.5. If $\epsilon = 0$ precompactness of $\chi\bar{F}^{(k)}_{\mathbf{r}}$ in $B^S_{P,q}(\mathbb{R}^n)$ follows from estimate (5.5) and the assumed precompactness of $g^{(k)}$ in $B^{s-1}_{p,q}(\mathbb{R}^n, E_2)$.*

REMARK 5.10. *We return to Remark 2.8. Assume that*

$$f^{(k)} \in B^{s,loc}_{p,q}(\mathbb{R}^n, E_1) \quad \textit{and} \quad g^{(k)} \in B^{s-\tau,loc}_{p,q}(\mathbb{R}^n, E_2)$$

are uniformly bounded, i.e. for all test functions $\chi \in \mathcal{D}(\mathbb{R}^n)$ the sequence $\chi f^{(k)}$ is bounded in $B^s_{p,q}(\mathbb{R}^n, E_1)$ etc. Then we have the following equality

$$\mathrm{div}_x(\mathbf{a}\chi f^{(k)}) = \chi g^{(k)} - f^{(k)}\mathrm{div}_x(\mathbf{a}\chi).$$

The RHS is uniformly bounded in $B^{s-\tau}_{p,q}(\mathbb{R}^n, E)$ with $E = E_1 + E_2$. From Theorem 2.5 we therefore conclude that the sequence $\chi\bar{f}^{(k)}$ is precompact in $B^S_{P,q}(\mathbb{R}^n)$. Hence it is also possible to choose $f^{(k)}$ and $g^{(k)}$ in local Besov spaces only.

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