The Radon transform and higher regularity of surfaces minimising a Finsler area

Joint work with Heiko von der Mosel

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Outline

1. Setting
   ➔ Finsler manifolds
   ➔ Busemann–Hausdorff area functional $\mathcal{A}_F$
   ➔ Plateau problem for $\mathcal{A}_F$

2. Hildebrandt/von der Mosel framework for Cartan functionals
   ➔ How does $\mathcal{A}_F$ fit into the Cartan theory?
   ➔ Existence of area minimisers

3. Higher regularity
   ➔ Ingredient: Perfect dominance functions
   ➔ Ingredient: Radon transform
Setting
Let $M^m$ and $N^n$ be smooth manifolds, $f: M \to N$ an immersion or embedding. Denote local coordinates on $M$ by $(u^1, \ldots, u^m)$.

Later:

- $m = 2$
- $M = D \subset \mathbb{R}^2$ unit disk
- $N = \mathbb{R}^n$
Finsler manifolds

Morally speaking:

Finsler metric = smooth family of smooth norms on tangent bundle.

This smoothened dodecahedron generates a Finsler metric on $\mathbb{R}^3$. 
A **Finsler metric** on $N$ is a function $F : TN \to [0, \infty)$ s.t.:

(F1) **Regularity**: $F \in C^k(TN \setminus \{0\}) \cap C(TN)$, $k \in \{2, 3, \ldots, +\infty\}$.

(F2) **Positive 1-homogeneity**: $F(x, tv) = tF(x, v)$ for all $t > 0$, $(x, v) \in TN$.

(F3) **Ellipticity**: For any $(x, v) \in TN \setminus \{0\}$ the “first fundamental form”

$$g^F_{x,v}(u, w) = \frac{d^2}{dtds}\bigg|_{t,s=0} \frac{1}{2} F(x, v + tu + sw)^2$$

describes a scalar product (which depends on $(x, v)$!).

A Finsler metric is **reversible** if $F(x, v) = F(x, -v)$ for all $(x, v) \in TN$. 

Busemann–Hausdorff area functional

Suppose $\bar{g}$ is an auxiliary Riemannian metric on the domain $M$. Let $\bar{F} = f^\# F$ be the pull-back Finsler metric on $M$.

Define the Busemann–Hausdorff area functional $A_F(f) = \int_M w_{\bar{F}, \bar{g}}$, where $\mu_{\bar{g}}$ is the Riemannian volume density of $\bar{g}$ and $w_{\bar{F}, \bar{g}}(p)$ is the weight function defined by $w_{\bar{F}, \bar{g}}(p) = H_{\mu_{\bar{g}}}(p)(B_{\bar{F}}(p)) = V_{\bar{g}}(p)$ for all $\bar{g}$.

Lemma: $A_F$ is independent of the choice of $\bar{g}$.
**Busemann–Hausdorff area functional**

Suppose $\tilde{g}$ is an auxiliary Riemannian metric on the domain $M$. Let $\tilde{F} = f^# F$ be the pull-back Finsler metric on $M$. Define the **Busemann–Hausdorff area** functional

$$\mathcal{A}_F(f) = \int_M w_{\tilde{F}, \tilde{g}} \mu_{\tilde{g}}$$

where $\mu_{\tilde{g}}$ is the Riemannian volume density of $\tilde{g}$ and $w_{\tilde{F}, \tilde{g}}$ is the weight function defined by

$$w_{\tilde{F}, \tilde{g}}(p) = \frac{\mathcal{H}^m_{\tilde{g}_p}(B_{\tilde{g}_p})}{\mathcal{H}^m_{\tilde{g}_p}(B_{\tilde{F}_p})} = \frac{\text{Vol}_{\tilde{g}}(\bullet)}{\text{Vol}_{\tilde{g}}(\square)}.$$

Here, $B_{\tilde{F}_p}$ and $B_{\tilde{g}_p}$ are the $\tilde{F}$- and $\tilde{g}$-unit balls in $T_p M$ and $\mathcal{H}^m_{\tilde{g}_p}$ is the Hausdorff measure in the Hilbert space $(T_p M, \tilde{g}_p)$.

Lemma: $\mathcal{A}_F$ is independent of the choice of $\tilde{g}$!
Busemann–Hausdorff area functional

Suppose $\bar{g}$ is an auxiliary Riemannian metric on the domain $M$. Let $\bar{F} = f^# F$ be the pull-back Finsler metric on $M$. Define the Busemann–Hausdorff area functional

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Lemma: $\mathcal{A}_F$ is independent of the choice of $\bar{g}$!
Busemann–Hausdorff area functional

Lemma: $\mathcal{A}_F$ is independent of the choice of $\tilde{g}$!

Sketch of proof:
Given Riemannian metrics $\tilde{g}, \tilde{h} \in \Sigma^2(T'M)$ there is a smooth bundle isomorphism $B: TM \rightarrow TM$ such that for all vector fields $X, Y \in \mathfrak{X}(M)$

$$\tilde{g}(X, Y) = \tilde{h}(BX, Y).$$
Busemann–Hausdorff area functional

**Lemma:** $\mathcal{A}_F$ is independent of the choice of $\bar{g}$!

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Given Riemannian metrics $\bar{g}, \bar{h} \in \Sigma^2(T'M)$ there is a smooth bundle isomorphism $B: TM \to TM$ such that for all vector fields $X, Y \in \mathfrak{X}(M)$

$$\bar{g}(X, Y) = \bar{h}(BX, Y).$$

The induced volume densities and Hausdorff measures transform as

$$\mu_{\bar{g}} = \sqrt{\det(B)} \mu_{\bar{h}},$$

$$\mathcal{H}^m_{\bar{g}} = \sqrt{\det(B)} \mathcal{H}^m_{\bar{h}}.$$

Thus,

$$\int_M \frac{\Omega_m}{\mathcal{H}^m_{\bar{g}}(B\bar{F})} \mu_{\bar{g}} = \int_M \frac{\Omega_m}{\mathcal{H}^m_{\bar{h}}(B\bar{F})} \mu_{\bar{h}}.$$
Plateau problem

Given a Jordan curve $\Gamma \subset \mathbb{R}^n$, is there a disk-type surface $f : D \rightarrow \mathbb{R}^n$ such that

$$\mathcal{A}_F(f) = \inf_{S(\Gamma)} \mathcal{A}_F$$

in the class of competing surfaces

$$S(\Gamma) = \{ f \in W^{1,2}(D; \mathbb{R}^n) : f|_{\partial D} \text{ parametrises } \Gamma \text{ weakly monotonically} \}$$
Hildebrandt/von der Mosel framework for Cartan functionals
If $f : M = D \rightarrow \mathbb{R}^n = N$ then a **Cartan functional** is of the form

$$C(f) = \int_D C(f, \frac{\partial f}{\partial u_1} \wedge \frac{\partial f}{\partial u_2}) \, du^1 du^2$$

where the **parametric integrand** $C \in C(\mathbb{R}^n \times \mathbb{R}^n)$ is

- positively 1-homogeneous,
- convex
- and of linear growth: $m_1|\sigma|_{\mathbb{R}^2} \leq C(x, \sigma) \leq m_2|\sigma|_{\mathbb{R}^2}$ in the second argument.
Remarks: Cartan functionals

\[ C(f) = \int_D C\left( f, \frac{\partial f}{\partial u^1} \wedge \frac{\partial f}{\partial u^2} \right) \, du^1 du^2 \]

- Positive 1-homogeneity of \( C \) corresponds to diffeomorphism-invariance of \( C \).
- Convexity and linear growth of \( C \) correspond to lower semicontinuity of \( C \).
- The three above conditions on \( C \) guarantee existence of \( C \)-minimisers.
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- Positive 1-homogeneity of \( C \) corresponds to diffeomorphism-invariance of \( C \).
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- The three above conditions on \( C \) guarantee existence of \( C \)-minimisers.
- Further conditions yield higher interior regularity (more on that later) and also regularity at the boundary.
- A lot of results Cartan functionals were investigated by [Hildebrandt, vdM ’99–’09].
As a Cartan functional

For some auxiliary Riemannian metric $g$ on the target $N$, consider the function $A_{F,g} : \bigwedge^m T N \to \mathbb{R}$ defined by

$$A_{F,g}(x, \sigma) = \frac{\Omega_m}{\mathcal{H}^m_{g_x}(B_{F_x} \cap \langle \sigma \rangle)} |\sigma|_{g_x, \bigwedge^m T_x N}$$

where $\langle \sigma \rangle = \{ v \in T_x N : \sigma \wedge v = 0 \}$ is the $m$-dimensional subspace of $T_x N$ spanned by $\sigma$. This is the Euclidean area integrand.
**$\mathcal{A}_F$ as a Cartan functional**

For some auxiliary Riemannian metric $g$ on the target $N$, consider the function $A_{F,g}: \bigwedge^m T N \rightarrow \mathbb{R}$ defined by

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where $\langle \sigma \rangle = \{v \in T_x N : \sigma \wedge v = 0\}$ is the $m$-dimensional subspace of $T_x N$ spanned by $\sigma$.

Recall that $\bar{F} = f^\# F$ and choose $\bar{g} = f^\# g$ in the definition of $\mathcal{A}_F$ to see that

$$\mathcal{A}_F(f) = \int_M \frac{\Omega_m}{\mathcal{H}^m_{\bar{g}}(B_{\bar{F}_p})} \mu_{\bar{g}}|_p$$

$f: (M, f^\# g) \rightarrow (f(M), \bar{g})$ isometric

$$= \int_M \frac{\Omega_m}{\mathcal{H}^m_{g_{f(p)}}(B_{F_{f(p)}} \cap df_p(T_p M))} \mu_{\bar{g}}|_p$$
\( \mathcal{A}_F \) as a Cartan functional

For some auxiliary Riemannian metric \( g \) on the target \( N \), consider the function \( A_{F,g} : \bigwedge_s^m TN \to \mathbb{R} \) defined by

\[
A_{F,g}(x, \sigma) = \frac{\Omega_m}{\mathcal{H}^m_{g_x}(B_{F_x} \cap \langle \sigma \rangle)} |\sigma|_{g_x, \bigwedge^m T_xN}
\]

where \( \langle \sigma \rangle = \{ v \in T_xN : \sigma \wedge v = 0 \} \) is the \( m \)-dimensional subspace of \( T_xN \) spanned by \( \sigma \).

Recall that \( \tilde{F} = f^#F \) and choose \( \tilde{g} = f^#g \) in the definition of \( \mathcal{A}_F \) to see that in local coordinates \( p = (u^1, \ldots, u^m) \)

\[\mathcal{A}_F(f) = \int_M \frac{\Omega_m}{\mathcal{H}^m_{\tilde{g}(p)}(B_{\tilde{F}_p})} \mu_{\tilde{g}}|p\]

\[f : (M, f^#g) \to (f(M), g) \text{ isometric} \]

\[= \int_M \frac{\Omega_m}{\mathcal{H}^m_{\tilde{g}(f(p))}(B_{F_f(p)} \cap df_p(T_pM))} \mu_{\tilde{g}}|p\]

\[= \int_M A_{F,g}(f(p), \left| \frac{\partial f}{\partial u^1} \right|_p \wedge \cdots \wedge \left| \frac{\partial f}{\partial u^m} \right|_p) \, du^1 \ldots du^m.\]
**$\mathcal{A}_F$ as a Cartan functional**

For some auxiliary Riemannian metric $g$ on the target $N$, consider the function $A_{F,g} : \bigwedge^m TN \to \mathbb{R}$ defined by

$$A_{F,g}(x, \sigma) = \frac{\Omega_m}{\mathcal{H}_g^m(B_{F_x} \cap \langle \sigma \rangle)} |\sigma|_{g_x, \bigwedge^m T_xN}$$

where $\langle \sigma \rangle = \{v \in T_xN : \sigma \wedge v = 0\}$ is the $m$-dimensional subspace of $T_xN$ spanned by $\sigma$.

Thus in local coordinates $p = (u^1, \ldots, u^m)$,

$$\mathcal{A}_F(f) = \int_M A_{F,g}(f, \frac{\partial f}{\partial u^1} \wedge \ldots \wedge \frac{\partial f}{\partial u^m}) \, du^1 \ldots du^m.$$

The integrand $A_{F,g} : \bigwedge^m TN \to \mathbb{R}$ is

- absolutely 1-homogeneous
- convexly extendible to $\bigwedge^m TN$

  - for $n = m + 1$ due to [Busemann ’49]
  - for $m = 2$, $n \in \mathbb{N}$ due to [Burago, Ivanov ’12]

in the second argument.
Theorem (Existence) [P, vdM ’17]

Let $M = D$, $F$ a Finsler and $g$ a Riemannian metric on $N = \mathbb{R}^n$ s.t.

$$c_F|v|_{g_x} \leq F(x, v) \leq C_F|v|_{g_x} \quad \text{for all } (x, v) \in T\mathbb{R}^n$$

and suppose $F$ satisfies a certain symmetrisation assumption†.

Then any given rectifiable Jordan curve $\Gamma \subset \mathbb{R}^n$ bounds a surface $f \in S(\Gamma)$ which minimises $\mathcal{A}_F$.

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†$F_{\text{sym}}(x, v) = (1/2(F(x, v)^{-m} + F(x, -v)^{-m}))^{-1/m}$ is also a Finsler metric
Theorem (Existence) [P, vdM ’17]

Let \( M = D \), \( F \) a Finsler and \( g \) a Riemannian metric on \( N = \mathbb{R}^n \) s.t.

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and suppose \( F \) satisfies a certain symmetrisation assumption†.

- Then any given rectifiable Jordan curve \( \Gamma \subset \mathbb{R}^n \) bounds a surface \( f \in S(\Gamma) \) which minimises \( \mathcal{A}_F \).
- In addition, the minimiser \( f \) is \( g \)-conformally parametrised on \( D \):

\[
f^\#g(\cdot, \cdot) = e^{2\lambda} \langle \cdot, \cdot \rangle_{\mathbb{R}^2} \quad \mathcal{H}^2 - \text{a.e. on } D.
\]

for some smooth function \( \lambda \).

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\[†F_{\text{sym}}(x, v) = (1/2(F(x, v)^{-m} + F(x, -v)^{-m}))^{-1/m} \text{ is also a Finsler metric}\]
Theorem (Existence) [P, vdM ’17]

Let \( M = D \), \( F \) a Finsler and \( g \) a Riemannian metric on \( N = \mathbb{R}^n \) s.t.
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c_F|v|_{g_x} \leq F(x, v) \leq C_F|v|_{g_x} \quad \text{for all } (x, v) \in T\mathbb{R}^n
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and suppose \( F \) satisfies a certain symmetrisation assumption\(^\dagger\).

- Then any given rectifiable Jordan curve \( \Gamma \subset \mathbb{R}^n \) bounds a surface \( f \in S(\Gamma) \) which minimises \( \mathcal{A}_F \).
- In addition, the minimiser \( f \) is \( g \)-conformally parametrised on \( D \):
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  f^\# g(\cdot, \cdot) = e^{2\lambda} \langle \cdot, \cdot \rangle_{\mathbb{R}^2} \quad \mathcal{H}^2 - \text{a.e. on } D.
  \]
  for some smooth function \( \lambda \).
- Furthermore, the minimiser \( f \) is of class
  \[
  C(\overline{D}; \mathbb{R}^n) \cap C^{0,\alpha}(D; \mathbb{R}^n) \cap W^{1,q}_{\text{loc}}(D; \mathbb{R}^n)
  \]
  for \( \alpha := (c_F/C_F)^2 \in (0, 1] \) and some \( q > 2 \).

\(^\dagger\) \( F_{\text{sym}}(x, v) = (1/2(F(x, v)^{-m} + F(x, -v)^{-m}))^{-1/m} \) is also a Finsler metric.
Some images

Minimising surface

Finsler unit ball

Numerical calculation due to Henrik Schumacher
Some images

Minimising surface

Finsler unit ball

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Some images

Minimising surface

Finsler unit ball
Some images

Minimising surface

Finsler unit ball
Some images

Minimising surface

Finsler unit ball
Higher regularity
Theorem [Overath, vdM ’13]

In codimension 1:

Let \( M = D, F \) a Finsler metric on \( N = \mathbb{R}^3 \) and \( g = \langle \cdot, \cdot \rangle_{\mathbb{R}^3} \). There is a constant \( c > 0 \) s.t.

- If

\[
\sup_{x \in \mathbb{R}^3} \left\| F(x, \cdot) - |\cdot|_{\mathbb{R}^3} \right\|_{C^2(S^2)} < c,
\]

then every conformal minimiser \( f \) of \( \mathcal{A}_F \) in \( S(\Gamma) \) is of class

\[
W^{2,2}_{\text{loc}}(D; \mathbb{R}^3) \cap C^{1,\mu}(D; \mathbb{R}^3)
\]

for some \( \mu \in (0, 1) \).
Theorem [P ’20 + ε]

In arbitrary codimension:

Let $M = D$, $F$ a Finsler metric on $N = \mathbb{R}^n$ and $g$ a Riemannian metric. There is a constant $c = c(g, n) > 0$ s.t.

- If

$$\sup_{x \in \mathbb{R}^n} \left\| F(x, \cdot) - |\cdot|_{g_x} \right\|_{C^2(\mathbb{S}^{n-1})} < c,$$

then every $g$-conformal minimiser $f$ of $\mathcal{A}_F$ in $\mathcal{S}(\Gamma)$ is of class

$$W^{2,2}_{\text{loc}}(D; \mathbb{R}^n) \cap C^{1,\mu}(D; \mathbb{R}^n)$$

for some $\mu \in (0, 1)$.
In arbitrary codimension:

Let $M = D$, $F$ a Finsler metric on $N$ and $g$ a Riemannian metric on $N$. There is a constant $c = c(g, n) > 0$ s.t.

If

$$\|F - g\|_{C^2(S(TN))} < c,$$

then every $g$-conformal minimiser $f$ of $\mathcal{A}_F$ in $S(\Gamma)$ is of class

$$W^{2,2}_{loc}(D; N) \cap C^{1,\mu}(D; N)$$

for some $\mu \in (0, 1)$. 
Theorem [P ’20+ε]

In arbitrary codimension:

Let $M = D$, $F$ a Finsler metric on $N = \mathbb{R}^n$ and $g$ a Riemannian metric. There is a constant $c = c(g, n) > 0$ s.t.

- If

$$\sup_{x \in \mathbb{R}^n} \left\| F(x, \cdot) - \cdot \right\|_{g_x} \left\| C^2(S^{n-1}_{g_x}) \right\| < c,$$

then every $g$-conformal minimiser $f$ of $\mathcal{A}_F$ in $\mathcal{S}(\Gamma)$ is of class

$$W^{2,2}_{loc}(D; \mathbb{R}^n) \cap C^{1,\mu}(D; \mathbb{R}^n)$$

for some $\mu \in (0, 1)$. 
Higher regularity

Ingredient: Perfect dominance functions
Theorem [Hildebrandt, vdM ’03]

Suppose \( C \) is a Cartan functional with parametric integrand \( C \), i.e. \( C \in C(\mathbb{R}^n \times \mathbb{R}^{\binom{n}{2}}) \) is

- positively 1-homogeneous,
- convex,
- and of linear growth.

Then every conformal minimiser \( f \) of \( C \) in \( S(\Gamma) \) is of class

\[
W^{2,2}_{loc}(D; \mathbb{R}^n) \cap C^{1,\mu}(D; \mathbb{R}^n)
\]

for some \( \mu \in (0, 1) \), if the parametric integrand \( C \) further

- is of class \( C^2(\mathbb{R}^n \times (\mathbb{R}^{\binom{n}{2}} \setminus \{0\})) \)
- and it possesses a perfect dominance function.
Definition: Dominance functions

A function \( D \in C(\mathbb{R}^n \times \mathbb{R}^{n\times 2}) \) is a **dominance function** for a parametric integrand \( C \in C(\mathbb{R}^n \times \mathbb{R}^{n\choose 2}) \) if

- \( C(x, p \wedge q) \leq D(x, P) \) for all \( x \in \mathbb{R}^n \) and \( P = (p, q) \in \mathbb{R}^{n\times 2} \),
- \( C(x, p \wedge q) = D(x, (p, q)) \) if and only if \( |p|_{\mathbb{R}^n}^2 = |q|_{\mathbb{R}^n}^2 \) and \( \langle p, q \rangle_{\mathbb{R}^n} = 0 \),

and in the second argument the dominance function \( D \) is

- positively \( 2 \)-homogeneous,
- and of **quadratic** growth: \( \mu_1 |P|_{\mathbb{R}^{n\times 2}}^2 \leq D(x, P) \leq \mu_2 |P|_{\mathbb{R}^{n\times 2}}^2 \).

A dominance function \( D \) is called **perfect** if \( D \in C^2(\mathbb{R}^n \times (\mathbb{R}^n \times \{0\})) \) and for \( R > 0 \) there is \( \lambda(R) > 0 \) s.t. for all \( |x|_{\mathbb{R}^n} < R \) the function \( P \mapsto D(x, P) - \lambda(R)|P|_{\mathbb{R}^{n\times 2}} \) is convex.
Definition: Dominance functions

A function $D \in C(\mathbb{R}^n \times \mathbb{R}^{n\times 2})$ is a **dominance function** for a parametric integrand $C \in C(\mathbb{R}^n \times \mathbb{R}^{(n^2)})$ if

- $C(x, p \wedge q) \leq D(x, P)$ for all $x \in \mathbb{R}^n$ and $P = (p, q) \in \mathbb{R}^{n\times 2}$,
- $C(x, p \wedge q) = D(x, (p, q))$ if and only if $|p|_{\mathbb{R}^n}^2 = |q|_{\mathbb{R}^n}^2$ and $\langle p, q \rangle_{\mathbb{R}^n} = 0$,

and in the second argument the dominance function $D$ is

- positively $2$-homogeneous,
- and of quadratic growth: $\mu_1 |P|_{\mathbb{R}^{n\times 2}}^2 \leq D(x, P) \leq \mu_2 |P|_{\mathbb{R}^{n\times 2}}^2$.

A dominance function $D$ is called **perfect** if

- $D \in C^2(\mathbb{R}^n \times (\mathbb{R}^{n\times 2} \setminus \{0\}))$ and
- for $R > 0$ there is $\lambda(R) > 0$ s.t. for all $|x|_{\mathbb{R}^n} < R$ the function

$$ P \mapsto D(x, P) - \lambda(R) \frac{1}{2} |P|_{\mathbb{R}^{n\times 2}}^2 $$

is convex.
Existence of perfect dominance functions

Theorem [Hildebrandt, vdM ’03]
Suppose $C \in C^2(\mathbb{R}^n \times (\mathbb{R}^{(\frac{n}{2}) \setminus \{0\}}))$ is positively 1-homogeneous, of linear growth (with constants $m_1$ and $m_2$) and uniformly elliptic, i.e. there is $\lambda > 0$ s.t. for all $x \in \mathbb{R}^n$ the function

$$\sigma \mapsto C(x, \sigma) - \lambda |\sigma|_{\mathbb{R}^{(\frac{n}{2})}}$$

is convex.

Then for every $k > k_0(m_1, m_2, \lambda)$ the new integrand

$$(x, \sigma) \mapsto C(x, \sigma) + k |\sigma|_{\mathbb{R}^{(\frac{n}{2})}}$$

possesses a perfect dominance function.

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$\dagger k_0(m_1, m_2, \lambda) = 2(m_2 - \min(\lambda, m_1/2))$
Existence of perfect dominance functions

**Theorem [Hildebrandt, vdM ’03]**
Suppose $C \in C^2(\mathbb{R}^n \times (\mathbb{R}^{(n)} \setminus \{0\}))$ is positively 1-homogeneous, of linear growth (with constants $m_1$ and $m_2$) and **uniformly elliptic**, i.e. there is $\lambda > 0$ s.t. for all $x \in \mathbb{R}^n$ the function

$$\sigma \mapsto C(x, \sigma) - \lambda |\sigma|_{\mathbb{R}^{(n)}}$$

is convex.

Then for every $k > k_0(m_1, m_2, \lambda)$ the new integrand

$$(x, \sigma) \mapsto C(x, \sigma) + k|\sigma|_{\mathbb{R}^{(n)}}$$

possesses a perfect dominance function.

**Corollary [Overath, vdM ’13]**
Suppose $C$ is as above and

$$\sup_{x \in \mathbb{R}^n} \left\| C(x, \cdot) - |\cdot|_{\mathbb{R}^{(n)}} \right\|_{C^2(\mathbb{S}^{(n)} - 1)} < \frac{1}{5}$$

then $C$ itself possesses a perfect dominance function.
Higher regularity

Ingredient: Radon transform
For $m = 2$ and $N = \mathbb{R}^n$:
Show that the integrand $A_{F,g} : \bigwedge^2 T\mathbb{R}^n \to \mathbb{R}$ of $\mathcal{A}_F$ satisfies

(i) $A_{F,g} \in C^2(\bigwedge^2 T\mathbb{R}^n \setminus \{0\})$,
(ii) $A_{F,g}$ is uniformly elliptic
(iii) and $\sup_{x \in \mathbb{R}^n} \left\| A_{F,g}(x, \cdot) - \cdot |g_x, \bigwedge^2 T_x \mathbb{R}^n \right\|_{C^2(S_{g_x}^{(n)} - 1)} < \frac{1}{5}$.

Compare condition (iii) to the smallness condition on

$$\sup_{x \in \mathbb{R}^n} \left\| F(x, \cdot) - \cdot |g_x \right\|_{C^2(S_{g_x}^{n - 1})}$$

in the main theorem.

These three goals can be achieved by using functional analytic properties of the so-called Radon transform.
Motivation: An easy calculation

Recall from earlier for $\sigma \in \bigwedge_s^m T_x N$:

$$A_{F,g}(x, \sigma) = \frac{\Omega_m}{\mathcal{H}_{g_x}^m(B_{F,x} \cap \langle \sigma \rangle)} |\sigma|_{g_x, \bigwedge_s^m T_x N}.$$

Using spherical coordinates in the Hilbert space $(\langle \sigma \rangle, g_x | \langle \sigma \rangle)$:

$$\mathcal{H}_{g_x}^m(B_{F,x} \cap \langle \sigma \rangle) = \int_{S^{m-1}_{g_x | \langle \sigma \rangle}} \int_0^\infty \chi_{B_{F,x} \cap \langle \sigma \rangle}(s\theta) \ s^{m-1} ds \ d\mathcal{H}_{g_x}^{m-1}(\theta).$$
Motivation: An easy calculation

Recall from earlier for $\sigma \in \bigwedge_s^m T_xN$:

$$A_{F,g}(x, \sigma) = \frac{\Omega_m}{\mathcal{H}^m_{g_x}(B_{F_x} \cap \langle \sigma \rangle)} |\sigma|_{g_x, \bigwedge^m T_xN}.$$  

Using spherical coordinates in the Hilbert space $(\langle \sigma \rangle, g_x|_{\langle \sigma \rangle})$:

$$\mathcal{H}^m_{g_x}(B_{F_x} \cap \langle \sigma \rangle) = \int_{S^{m-1}_{g_x|_{\langle \sigma \rangle}}} \int_0^\infty \chi_{B_{F_x} \cap \langle \sigma \rangle}(s\theta) \ s^{m-1} ds \ dH^{m-1}_{g_x}(\theta).$$

Note that $s\theta \in B_{F_x}$ if and only if $s < F(x, \theta)^{-1}$.
Motivation: An easy calculation

Recall from earlier for $\sigma \in \bigwedge^m_s T_xN$:

$$A_{F,g}(x, \sigma) = \frac{\Omega_m}{\mathcal{H}^m_{g_x}(B_{F_x} \cap \langle \sigma \rangle)} |\sigma|_{g_x, \bigwedge^m T_xN}.$$ 

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Note that $s\theta \in B_{F_x}$ if and only if $s < F(x, \theta)^{-1}$ and hence

$$\frac{\mathcal{H}_g^m(B_{F_x} \cap \langle \sigma \rangle)}{\Omega_m} = \frac{1}{m\Omega_m} \int_{\mathbb{S}^{m-1}_{g_x |\langle \sigma \rangle}} F(x, \cdot)^{-m} d\mathcal{H}_g^{m-1}.$$
Motivation: An easy calculation

Recall from earlier for \( \sigma \in \bigwedge_s^m T_xN \):

\[
A_{F,g}(x, \sigma) = \frac{\Omega_m}{\mathcal{H}_g^m(B_F \cap \langle \sigma \rangle)} |\sigma|_{g_x, \bigwedge^m T_xN}.
\]

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Note that \( s\theta \in B_F \) if and only if \( s < F(x, \theta)^{-1} \) and hence

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\frac{\mathcal{H}_g^m(B_F \cap \langle \sigma \rangle)}{\Omega_m} = \frac{1}{m\Omega_m} \int_{S^{m-1}} F(x, \cdot)^{-m} d\mathcal{H}_g^{m-1}
\]

\((\langle \sigma \rangle, g_x | \langle \sigma \rangle) \hookrightarrow (T_xM, g_x)\) isometrically

\[
= \frac{1}{\omega_{m-1}} \int_{S^{n-1}} F(x, \cdot)^{-m} d\mathcal{H}_g^{m-1}.
\]
The Radon transform: Definition

Let $\text{Gr}_m(\mathbb{R}^n)$ denote the Grassmannian, the set of $m$-dimensional subspaces of $\mathbb{R}^n$.

The linear operator $\mathcal{R}: C(S^{n-1}) \to C(\text{Gr}_m(\mathbb{R}^n))$ defined by

$$\mathcal{R}[f](E) = \frac{1}{\omega_{m-1}} \int_{S^{n-1} \cap E} f \, d\mathcal{H}^{m-1}$$

is called the **Radon transform** on $\mathbb{R}^n$. 
The Radon transform: Definition

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$$

is called the **Radon transform** on $\mathbb{R}^n$.

This can be lifted to a linear operator $\mathcal{R}_g$ on the sphere bundle and the Grassmann bundle of $(N, g)$.

**Corollary**

The area integrand $A_{F,g}$ satisfies

$$
A_{F,g}(x, \sigma) = \left( \mathcal{R}_g[F(x, \cdot)^{-m}](\langle \sigma \rangle) \right)^{-1} |\sigma|_{g_x, \wedge^m T_x N}.
$$
The Radon transform: Properties

- Downside to using $A_{F,g}$:
  In high codimension the base space $\bigwedge^m_s TN$ is a bundle of cones and difficult to handle.

- Benefits of the Radon transform:
  Base spaces $\mathbb{S}^{n-1}$ and $\text{Gr}_m(\mathbb{R}^n)$ are homogeneous $O(n)$-spaces with a lot of algebraic and geometric structure.

- By the Plücker embedding both base spaces are related via

\[
\text{Gr}_m(\mathbb{R}^n) \cong (\mathbb{S}(\bigwedge^m \mathbb{R}^n) \cap \bigwedge^m_s (\mathbb{R}^n))/\mathbb{Z}_2.
\]
The Radon transform: Properties

For $E \in \text{Gr}_m(\mathbb{R}^n)$:

$$\mathcal{R}[f](E) = \frac{1}{\omega_{m-1}} \int_{\mathbb{S}^{n-1} \cap E} f \, d\mathcal{H}^{m-1}.$$ 

**Lemma (O(n)-Equivariance)**

For any $f \in C(\mathbb{S}^{n-1})$ and $Q \in O(n)$:

$$\mathcal{R}[f] \circ Q = \mathcal{R}[f \circ Q].$$
The Radon transform: Properties

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Lemma ($O(n)$-Equivariance)

For any $f \in C(S^{n-1})$ and $Q \in O(n)$:

$$\mathcal{R}[f] \circ Q = \mathcal{R}[f \circ Q].$$

Taking derivative at $Q = \text{Id}_{\mathbb{R}^n}$ leads to:

**Theorem (Differentiability)**

Suppose $f \in C^1(S^{n-1})$. Then $\mathcal{R}[f] \in C^1(\text{Gr}_m(\mathbb{R}^n))$ and for all $X \in \mathfrak{o}(n)$:

$$d(\mathcal{R}[f])(\mathcal{K}_X) = \mathcal{R}[df(\mathcal{K}_X)].$$

Fundamental vector fields generated by $O(n)$-actions
The Radon transform: Properties

**Theorem (Differentiability)**
Suppose $f \in C^1(S^{n-1})$. Then $R[f] \in C^1(\text{Gr}_m(\mathbb{R}^n))$ and for all $X \in \mathcal{V}(n)$:

$$d(R[f])(K_X) = R[df(K_X)].$$

**Theorem (Higher differentiability)**
For any $k \geq 0$ the Radon transform

$$R : C^k(S^{n-1}) \to C^k(\text{Gr}_m(\mathbb{R}^n))$$

is a continuous operator with explicit bounds for its operator norm.
The Radon transform: Properties

**Theorem (Differentiability)**
Suppose $f \in C^1(\mathbb{S}^{n-1})$. Then $\mathcal{R}[f] \in C^1(\text{Gr}_m(\mathbb{R}^n))$ and for all $X \in \mathfrak{o}(n)$:

$$d(\mathcal{R}[f])(\mathcal{K}_X) = \mathcal{R}[df(\mathcal{K}_X)].$$

**Theorem (Higher differentiability)**
For any $k \geq 0$ the Radon transform

$$\mathcal{R} : C^k(\mathbb{S}^{n-1}) \rightarrow C^k(\text{Gr}_m(\mathbb{R}^n))$$

is a continuous operator with explicit bounds for its operator norm.

**Theorem (Invertibility, [Helgason, ’90])**
The Radon transform is an invertible operator when restricted to even functions on the sphere.

†Helgason gives an explicit inversion formula
We needed to prove three properties for the area integrand $A_{F,g}$ to apply the theorems of Hildebrandt/von der Mosel:

(i) $A_{F,g} \in C^2(\bigwedge^2 S^2 T\mathbb{R}^n \setminus \{0\})$,
   → Use higher differentiability of $\mathcal{R}$

(ii) $A_{F,g}$ is uniformly elliptic
    → Use invertibility of $\mathcal{R}$

(iii) and $\sup_{x \in \mathbb{R}^n} \left\| A_{F,g}(x, \cdot) - |\cdot|_{g_x, \bigwedge^2 T_x \mathbb{R}^n} \right\|_{C^2(S_x(g_x)^{-1}} < \frac{1}{5}$.
    → Use operator norm bounds for $\mathcal{R}$
The Radon transform: Summary

- We needed to prove three properties for the area integrand $A_{F,g}$ to apply the theorems of Hildebrandt/von der Mosel:
  
  1. $A_{F,g} \in C^2(\bigwedge^2_s T\mathbb{R}^n \setminus \{0\})$,
      - Use higher differentiability of $R$
  2. $A_{F,g}$ is uniformly elliptic
      - Use invertibility of $R$
  3. and $\sup_{x \in \mathbb{R}^n} \left\| A_{F,g}(x, \cdot) - |\cdot|_{g_x, \bigwedge^2 T_x \mathbb{R}^n} \right\|_{C^2(S^{(S^2(n))^{-1}})} < \frac{1}{5}$.
      - Use operator norm bounds for $R$

- The main difficulty in high codimension is that $\bigwedge^m_s T N$ is not a vector bundle!

- Use the Radon transform and exploit the nice algebraic structure of the base spaces!
Theorem [P ’20+ε]

Let $M = D$, $F$ a Finsler metric on $N = \mathbb{R}^n$ and $g$ a Riemannian metric. There is a constant $c = c(g, n) > 0$ s.t.

- If

$$\sup_{x \in \mathbb{R}^n} \left\| F(x, \cdot) - | \cdot |_{g_x} \right\|_{C^2(S_{g_x}^{n-1})} < c,$$

then every $g$-conformal minimiser $f$ of $\mathcal{A}_F$ in $\mathcal{S}(\Gamma)$ is of class

$$W^{2,2}_{\text{loc}}(D; \mathbb{R}^n) \cap C^{1,\mu}(D; \mathbb{R}^n)$$

for some $\mu \in (0, 1)$. 

S. Pistre (and H. von der Mosel) | The Radon transform and higher regularity of surfaces minimising a Finsler area
Thank you for your attention!

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