

A speed preserving Hilbert gradient flow for generalized integral Menger curvature

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Online Seminar “Geometric Analysis”
April 14 2020

Outline

Knots and Knot Energies

Short Time Existence

Long Time Existence via Projected Gradients

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Long Time Existence via Projected Gradients

What is a Knot?

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A **knot** is an embedding $\gamma : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^n$.

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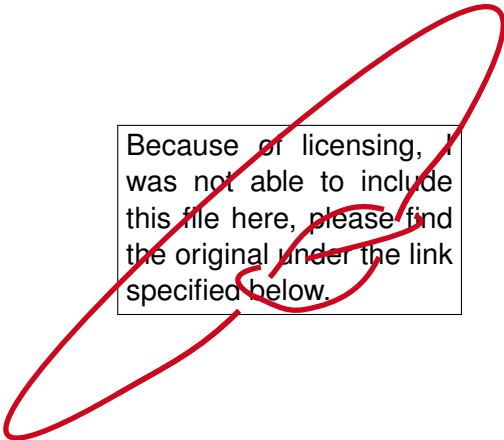
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Knot Classes

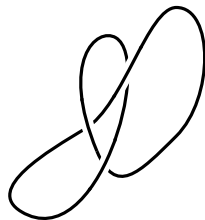
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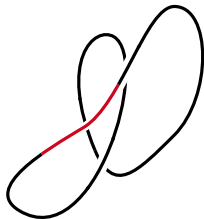


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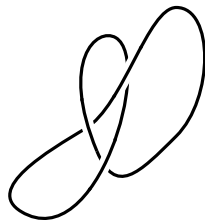


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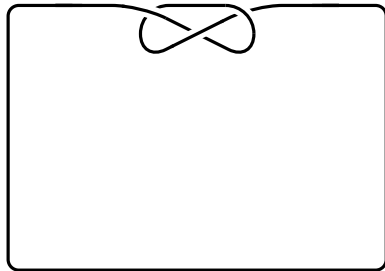
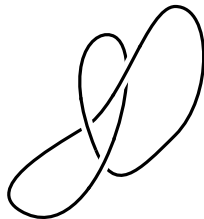


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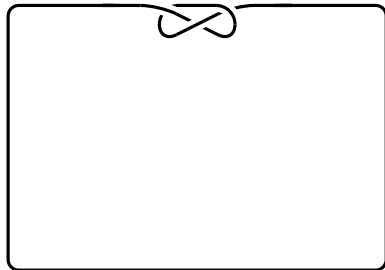
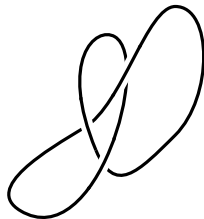


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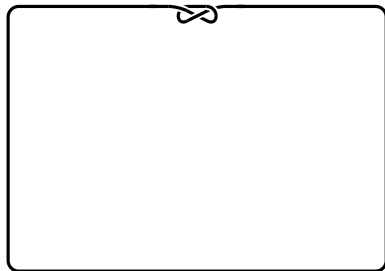
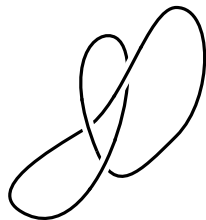


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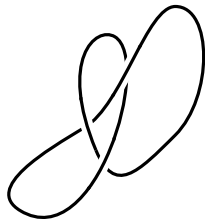


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How to Tell Knots Apart?

Geometric knot theory: **Knot Energies**

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Image: Pawel Strzelecki, Heiko von der Mosel - Menger curvature as a knot energy, Figure 10, Physics Reports

How to Tell Knots Apart?

Geometric knot theory: **Knot Energies**

A functional that

- is finite on all smooth knots
- approaches infinity when the arguments uniformly approach a curve with a self-intersection
- is bounded below

is called a **knot energy**.

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Tight knot energies penalize pulling knots tight.

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Generalized integral Menger curvature [Blatt, Reiter 2015]

Let $p, q > 0$, $\gamma \in \text{AC}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Then, the **generalized integral Menger curvature** is defined as

$$\text{intM}^{(p,q)}(\gamma) := \iiint_{\gamma(\mathbb{R}/\mathbb{Z})^3} \frac{1}{R^{(p,q)}(x, y, z)} dH^1(x) dH^1(y) dH^1(z)$$

where

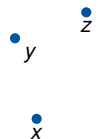
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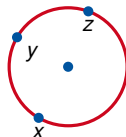
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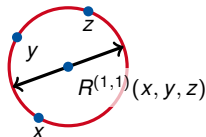
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The Space Associated with $\text{intM}^{(p,q)}$

For $p \in (\frac{2}{3}q + 1, q + \frac{2}{3})$, $q > 1$, $\text{intM}^{(p,q)}$ is associated with $W^{\frac{3p-2}{q}-1, q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, a fractional Sobolev space. [Blatt, Reiter 2015]

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$W^{k+s,q} \subseteq W^{k,q}$ is characterized by

$$[\gamma^{(k)}] := \left(\int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|\gamma^{(k)}(u+w) - \gamma^{(k)}(u)|^q}{|w|^{1+s \cdot q}} dw du \right)^{\frac{1}{q}} < \infty$$

$[\gamma^{(k)}]$ is called **Gagliardo** or **Slobodeckii seminorm**.

Abbreviations

For $q = 2$, we have the Hilbert space $\mathcal{H} := W^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.
Set

- $U := W_{\text{ir}}^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, the subset of injective and **regular** (i.e. positive speed) curves and
- $E_p := \text{int}M^{(p,2)}$.

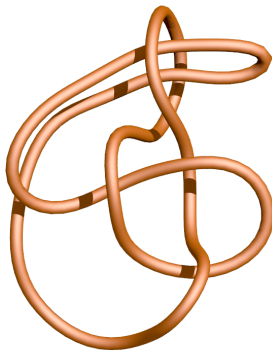
The Problem

We want a solution to

$$\begin{aligned}\frac{d}{dt}\gamma(\cdot, t) &= -\nabla E_p(\gamma(\cdot, t)), \\ \gamma(\cdot, 0) &= \gamma_0 \in U.\end{aligned}\tag{Flow}$$

Such a flow would preserve the knot class and decrease E_p as quickly as possible.

The Flow for $E_{2.6\bar{3}}$



Main Theorem

Let $\gamma_0 \in U$.

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Additionally, we have that

$$(t \mapsto \gamma(\cdot, t)) \in C^{1,1}([0, \infty), \mathcal{H})$$

and, for all $x \in \mathbb{R}/\mathbb{Z}$, $t \in [0, \infty)$,

$$\left| \frac{d}{dx}\gamma(x, t) \right| = \left| \frac{d}{dx}\gamma_0(x) \right|.$$

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Knots and Knot Energies

Short Time Existence

Long Time Existence via Projected Gradients

A Quantified Picard-Lindelöf Theorem

Picard-Lindelöf Theorem

Let $A : B_r(z) \rightarrow \mathcal{H}$ be Lipschitz continuous. Then, the differential equation

$$\dot{u}(t) = A(u(t))$$

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has a unique short-time solution.

Existence time depends only on

- upper bound M for A ,
- Lipschitz Radius r :

$$T \geq \frac{r}{M}$$

[cf. Cartan 1967].

Short Time Existence

E_p is $C_{\text{loc}}^{1,1}$ with

$$r_{E_p} = r_{E_p}\left(p, \overset{\nearrow}{\underset{\uparrow}{1}}{\text{BiLip}(\gamma_0)}\right), \quad M_{E_p} = M_{E_p}\left(p, \overset{\nearrow}{\underset{\uparrow}{1}}{\text{BiLip}(\gamma_0)}, [\overset{\nearrow}{\gamma'_0}]\right)$$

Here, $\text{BiLip}(\gamma) := \inf_{\substack{x, y \in \mathbb{R}/\mathbb{Z} \\ x \neq y}} \frac{|\gamma(x) - \gamma(y)|}{|x - y|_{\mathbb{R}/\mathbb{Z}}}$.

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\Rightarrow Flow from γ_0 exists at least up to time

$$T = T \left(p, \overset{\swarrow}{\underset{\searrow}{1}} \text{BiLip}(\gamma_0), [\overset{\swarrow}{\underset{\searrow}{\gamma'_0}}] \right).$$

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Controlling Existence Times

For γ parametrized by arc-length,

$$\frac{1}{\text{BiLip}(\gamma)} \leq C(p, E_p(\gamma))$$

and

$$[\gamma'] \leq C(p, E_p(\gamma)),$$

[Blatt, Reiter 2015].

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Projected Gradients [Neuberger 1997]

$\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces, $\tilde{U} \subseteq \mathcal{H}_1$ open, $E : \tilde{U} \rightarrow \mathbb{R}$, $S : \tilde{U} \rightarrow \mathcal{H}_2$, both Fréchet differentiable.

$$\frac{d}{dt}u(t) = -\nabla_S E(u(t)) := -P_{\mathcal{N}(DS(u(t)))}\nabla E(u(t)).$$

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Hence,

$$S(u(t)) = S(u(0))$$

for all t where $u(t)$ exists.

Existence for the new RHS

Is the new RHS locally Lipschitz?

Yes, if $S \in C_{\text{loc}}^{1,1}$ and $DS(x_0)$ has a right-inverse Y .

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The Lipschitz radius around $x_0 \in \tilde{U}$ is

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Thus, the existence time at x_0 is at least

$$T(M_E(x_0), \|DS(x_0)\|, \|Y(x_0)\|, L_{DS}(x_0), r_E(x_0)).$$

The Logarithmic Strain Constraint

From now on, let

$$S : U \rightarrow W^{\frac{3}{2}p-3,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}),$$
$$\gamma \mapsto (x \mapsto \ln(|\gamma'(x)|)),$$

the **logarithmic strain**. [Scholtes, Schumacher, Wardetzky 2019]

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New problem:

$$\begin{aligned} \frac{d}{dt}\gamma(\cdot, t) &= -\nabla_S E_p(\gamma(\cdot, t)), \\ \gamma(\cdot, 0) &= \gamma_0 \in U. \end{aligned} \tag{Flow}_S$$

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Supplementary Facts

- $\mathcal{L}(\gamma(\cdot, t)) = \mathcal{L}(\gamma_0)$ and $\int_{\mathbb{R}/\mathbb{Z}} \gamma(x, t) dx = \int_{\mathbb{R}/\mathbb{Z}} \gamma_0(x) dx$ for all $t > 0$.
- $t \mapsto E_p(\gamma(\cdot, t))$ is either strictly decreasing or constant [Neuberger 1997].
- There is a quicker way of proving long time existence, but it does not control the Lipschitz constant of the time-derivative.
- The same techniques may be applied to the generalized tangent-point energy $TP^{(p,2)}$, a related knot energy.

Summary

- The gradient flow for $\text{intM}^{(p,2)}$ exists for a finite time if $\gamma_0 \in W_{\text{ir}}^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.
- A slightly modified flow for $\text{intM}^{(p,2)}$ exists for all times if $\gamma_0 \in W_{\text{ir}}^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

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Outlook

- Does the modified flow converge?
- What about the non-Hilbertian setting?

Thank you for your time and
attention!

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