Long-Time Existence of a Parametrization-Preserving Gradient Flow for Generalized Integral Menger Curvature in the Hilbert Case

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Winter School Gradient Flows and Variational Methods in PDEs

Outline

Knots and Knot Energies

Short Time Existence

Long Time Existence via Projected Gradients

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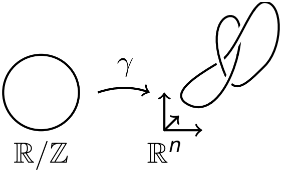
Knots and Knot Energies



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$\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$.

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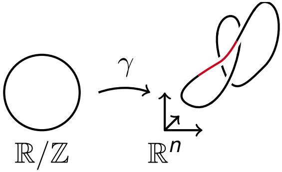

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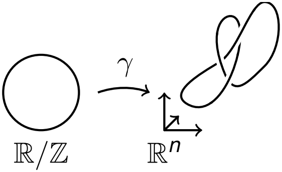

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A functional that

- is finite on all smooth knots
- approaches infinity when the arguments uniformly approach a curve with a self-intersection
- is bounded below

is called a knot energy.

Generalized Integral Menger Curvature [Blatt, Reiter 2015]

Let $p, q > 0$, $\gamma \in AC(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Then, the generalized integral Menger curvature is defined as

$$\mathrm{intM}^{(p,q)}(\gamma) := \iiint_{\gamma(\mathbb{R}/\mathbb{Z})^3} \frac{1}{R^{(p,q)}(x,y,z)} dH^1(x)\, dH^1(y)\, dH^1(z)$$

where

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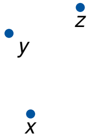

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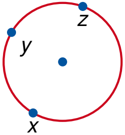

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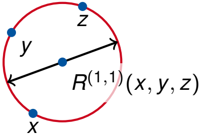

The Space Associated with $\mathrm{intM}^{(p,q)}$

For $p \in (\frac{2}{3}q + 1, q + \frac{2}{3})$, $q > 1$, $\mathrm{intM}^{(p,q)}$ is associated with $W^{\frac{3p-2}{q}-1,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, a fractional Sobolev space.

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For $q = 2$, we have the Hilbert space $\mathcal{H} := W^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Set

- $U := W_{\mathrm{ir}}^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, the subset of injective and regular (i.e. positive speed) curves and
- $E_p := \mathrm{intM}^{(p,2)}$.

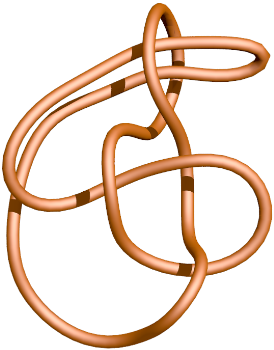
The Problem

We want a solution to

$$\frac{d}{dt}\gamma(\cdot, t) = -\nabla E_p(\gamma(\cdot, t)),$$
$$\gamma(\cdot, 0) = \gamma_0 \in U. \tag{Flow}$$

Such a flow would preserve the knot class.

The Flow for $E_{2.6\overline{3}}$



Outline

A Picard-Lindelöf Theorem for Banach Spaces

Existence time for Picard-Lindelöf depends only on

- upper bound M for RHS,
- Lipschitz Radius R:

$$T \geq \frac{R}{M}$$

[Cartan 1967].

Short Time Existence

E_p is $C^{1,1}_{\text{loc}}$ with nice constants $\Rightarrow$ Flow exists at least up to time

$$T = T\left(p, \frac{1}{\operatorname{BiLip}(\gamma_0)}, [\gamma_0']\right).$$

Here, BiLip(γ) $:= \inf_{\substack{x,y \in \mathbb{R}/\mathbb{Z} \\ x \neq y}} \frac{|\gamma(x) - \gamma(y)|}{|x-y|_{\mathbb{R}/\mathbb{Z}}}$ and

$$[\gamma'] := \left(\int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|\gamma'(u+w) - \gamma'(u)|^2}{|w|^{3p-5}} dw \, du \right)^{\frac{1}{2}}$$

Controlling Existence Times

For γ parametrized by arc-length,

$$\frac{1}{\mathrm{BiLip}(\gamma)} \leq C(p, \nearrow E_p(\gamma))$$

and

$$[\gamma'] \leq C(p, \nearrow E_p(\gamma)),$$

[Blatt, Reiter 2015].

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Projected Gradients [Neuberger 1997]

$\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces, $\widetilde{U} \subseteq \mathcal{H}_1$ open, $E : \widetilde{U} \to \mathbb{R}$, $\mathfrak{S} : \widetilde{U} \to \mathcal{H}_2$, both Fréchet differentiable.

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Hence,

$$\mathfrak{S}(u(t)) = \mathfrak{S}(u(0))$$

for all t where $u(t)$ exists.

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The Lipschitz radius around $x_0 \in \widetilde{U}$ is

$$r_{\text{RHS}} = r_{\text{RHS}}\big(\overset{\searrow}{\|D\mathfrak{S}(x_0)\|}, \overset{\searrow}{\|R(x_0)\|}, \overset{\searrow}{L_{D\mathfrak{S}}(x_0)}, \overset{\nearrow}{r_{\nabla E}(x_0)}\big).$$

where $R(x_0)$ is a right-inverse to $D\mathfrak{S}(x_0)$.

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where $R(x_0)$ is a right-inverse to $D\mathfrak{S}(x_0)$. Thus, the existence time at x_0 is at least

$$T(\overset{\searrow}{M(x_0)}, \overset{\searrow}{\|D\mathfrak{S}(x_0)\|}, \overset{\searrow}{\|R(x_0)\|}, \overset{\searrow}{L_{D\mathfrak{S}}(x_0)}, \overset{\nearrow}{r_{\nabla E}(x_0)}).$$

The Logarithmic Strain as Constraint

From now on, let

$$\begin{aligned} \mathfrak{S} : U &\to W^{\frac{3}{2}p-3,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}), \\ \gamma &\mapsto \left(x \mapsto \ln\left(\left|\gamma'(x)\right|\right)\right), \end{aligned}$$

the logarithmic strain ([Scholtes, Schumacher, Wardetzky 2019]).

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the logarithmic strain ([Scholtes, Schumacher, Wardetzky 2019]). New problem:

$$\begin{aligned} \frac{d}{dt}\gamma(\cdot, t) &= -\nabla_{\mathfrak{S}} E_p(\gamma(\cdot, t)), \\ \gamma(\cdot, 0) &= \gamma_0 \in U. \end{aligned} \qquad (\mathrm{Flow}_{\mathfrak{S}})$$

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Additionally, we have that

$$\left(t \mapsto \gamma(\cdot, t)\right) \in C^{1,1}\left([0, \infty), \mathcal{H}\right)$$

and, for all $x \in \mathbb{R}/\mathbb{Z}$, $t \in [0, \infty)$,

$$\left| \frac{d}{dx}\gamma(x, t) \right| = \left| \frac{d}{dx}\gamma_0(x) \right|.$$

Thank you for your time and attention!

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