A speed preserving Hilbert gradient flow for generalized integral Menger curvature

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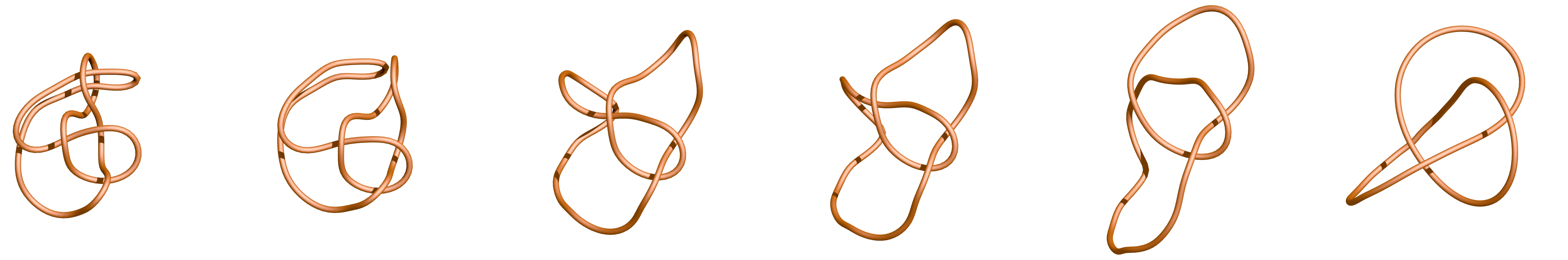

Figure 1: *Numerical version of a gradient flow for* $\operatorname{intM}^{(2.6\bar{3},2)}$

1. Generalized Integral Menger Curvature [Blatt, Reiter 2015]

The generalized integral Menger curvature $\operatorname{intM}^{(p,q)}(\gamma)$ is defined as

$$\iiint_{(\mathbb{R}/\mathbb{Z})^3} \frac{|\gamma'(u_1)|\,|\gamma'(u_2)|\,|\gamma'(u_3)|}{R^{(p,q)}(\gamma(u_1),\gamma(u_2),\gamma(u_3))} du_1\,du_2\,du_3$$

where

$$R^{(p,q)}(x,y,z) := \frac{(|z-x|\,|y-x|\,|z-y|)^p}{|(z-x)\wedge(y-x)|^q}.$$

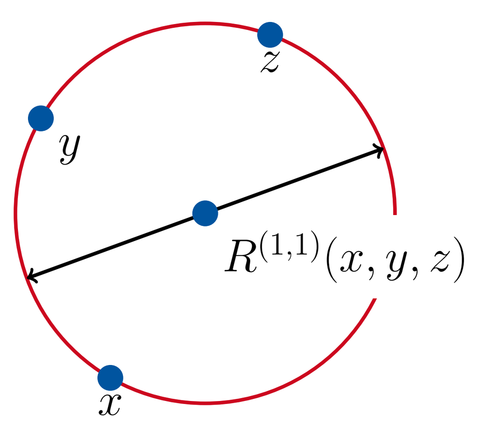

Figure 2: $R^{(1,1)}$ *is simply the diameter of the unique circle through* x, y *and* z.

We have that $\operatorname{intM}^{(p,p)} = 2^p \mathcal{M}_p$, the integral Menger curvature.
For injective curves $\gamma \in C^1(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ parametrized by arc-length,

$$\operatorname{intM}^{(p,q)}(\gamma) < \infty \quad \Leftrightarrow \quad \gamma \in W^{\frac{3p-2}{q}-1,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$$

with $q > 1$, $p \in \left(\frac{2}{3}q+1, q+\frac{2}{3}\right)$. We work in the Hilbert space setting $q = 2$, requiring $p \in \left(\frac{7}{3},\frac{8}{3}\right)$.

2. Existence Times for the Gradient Flow

We want existence of the gradient flow, i.e. a solution to

$$\frac{d}{dt}\gamma(\cdot,t) = -\nabla \operatorname{intM}^{(p,2)}(\gamma(\cdot,t))$$
$$\gamma(\cdot,0) = \gamma_0 \in W^{\frac{3}{2}p-2,2}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3),$$

the index ir meaning injective curves which are regular, i.e. $|\gamma_0'(\cdot)| \geq c > 0$. The right hand side is locally Lipschitz continuous and we have bounds for both it and the radius of the balls where Lipschitz continuity holds. With these bounds, we obtain a minimal existence time which depends non-increasingly on

$$\frac{1}{\operatorname{BiLip}(\gamma_0)} := \sup_{x,y\in\mathbb{R}/\mathbb{Z}} \frac{|x-y|_{\mathbb{R}/\mathbb{Z}}}{|\gamma_0(x)-\gamma_0(y)|}$$

and the Gagliardo-semi-norm

$$[\gamma_0'] := [\gamma_0']_{W^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)}.$$

3. Controlling Bi-Lipschitz Constant and Semi-norm via $\operatorname{intM}^{(p,q)}$

For curves parametrized by arc-length, Blatt and Reiter give estimates for both quantities. We have, for non-decreasing $\lambda,\mu : [0,\infty) \to [0,\infty)$, that

$$\frac{1}{\operatorname{BiLip}(\gamma)} \leq \mu(\operatorname{intM}^{(p,2)}(\gamma))$$

and

$$[\gamma'] \leq \lambda(\operatorname{intM}^{(p,2)}(\gamma)).$$

We can extend these estimates to curves that have a fixed parametrization. Thus, if we have a flow that preserves parametrization, we can find a uniform lower bound for the existence time of each new flow starting at the end of the last one. This would mean that we have long time existence.

4. Preserving Constraints via Projected Gradients [Neuberger 1997]

Consider Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, an open set $U \subseteq \mathcal{H}_1$ and two Fréchet differentiable maps $E : U \to \mathbb{R}$, the energy, and $\mathfrak{G} : U \to \mathcal{H}_2$, the constraint. Let $P_{\mathcal{N}(A)}$ denote the orthogonal projection onto the null-space of an operator A. Assuming a classical solution to

$$\frac{d}{dt}u(t) = -\nabla_{\mathfrak{G}} E(u(t))$$
$$:= -P_{\mathcal{N}(D\mathfrak{G}(u(t)))}\nabla E(u(t))$$

exists on $[0,T]$, we have

$$\mathfrak{G}(u(t)) = \mathfrak{G}(u(0))$$

for all $t \in [0,T]$.

5. Local Lipschitz Continuity of the Projected Gradient

If the constraint is $C^{1,1}_{\mathrm{loc}}$ with pointwise surjective differential, the right hand side is locally Lipschitz and the flow starting at $x_0 \in U$ exists at least up to time

$$T(M(x_0), \|D\mathfrak{G}(x_0)\|, \|R_{x_0}\|, L_{D\mathfrak{G}}(x_0), r_{\nabla E}(x_0)),$$

where R_{x_0} is a right-inverse to $D\mathfrak{G}(x_0)$, $L_{D\mathfrak{G}}$ is the Lipschitz constant of $D\mathfrak{G}$ and $M(x_0)$ and $r_{\nabla E}(x_0)$ are an upper bound and a Lipschitz radius for ∇E. T is non-decreasing in its last argument and non-increasing in all the others.

6. A Constraint that Preserves the Parametrization [Scholtes, Schumacher, Wardetzky 2019]

We define $\mathfrak{G}$ as the logarithmic strain:

$$\mathfrak{G} : W^{\frac{3}{2}p-2,2}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3) \to W^{\frac{3}{2}p-3,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}),$$
$$\gamma \mapsto \ln(|\gamma'(\cdot)|)$$

$\mathfrak{G}$ is Fréchet differentiable and $D\mathfrak{G}$ is locally Lipschitz continuous.
Also, $D\mathfrak{G}(\gamma)$ is surjective with right inverse R_γ and

$$\|D\mathfrak{G}(\gamma)\|, \|R_\gamma\|, L_{D\mathfrak{G}}(\gamma) \leq C\left([\gamma'], \frac{1}{\operatorname{BiLip}(\gamma)}\right),$$

where C is non-decreasing in its components.

7. Minimal Existence Time for the Projected Gradient Flow

There is a unique classical solution to

$$\frac{d}{dt}\gamma(\cdot,t) = -\nabla_{\mathfrak{G}} \operatorname{intM}^{(p,2)}(\gamma(\cdot,t)) \qquad (*)$$
$$\gamma(\cdot,0) = \gamma_0 \in W^{\frac{3}{2}p-2,2}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)$$

up to time

$$T = T\left([\gamma_0'], \frac{1}{\operatorname{BiLip}(\gamma_0)}\right) > 0.$$

T is non-increasing in its components. With the estimates by Blatt and Reiter, we may instead use the lower bound

$$T = T(\operatorname{intM}^{(p,2)}(\gamma_0))$$

which is non-increasing.

8. Continuing the Flow

Since

$$\frac{d}{dt}\operatorname{intM}^{(p,2)}(\gamma(\cdot,t)) \leq 0,$$

we can restart the flow at time $T_0 := T(\operatorname{intM}^{(p,2)}(\gamma_0))$. This yields a new existence time $T(\operatorname{intM}^{(p,2)}(\gamma(\cdot,T_0)))$ which is at least as long as T_0 itself. Iterating this argument yields long time existence.

9. Main Theorem: Long Time Existence

There is a unique classical solution to $(*)$ which exists for all positive times. We have that

$$(t \mapsto \gamma(\cdot,t)) \in C^{1,1}\left([0,\infty), W^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^3)\right)$$

and, for all $t \geq 0$,

$$\left|\frac{d}{dx}\gamma(x,t)\right| = \left|\frac{d}{dx}\gamma_0(x)\right|.$$

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