A speed preserving Hilbert gradient flow for generalized integral Menger curvature

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Outline

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Knots and Knot Energies

Short Time Existence

Long Time Existence via Projected Gradients

What is a Knot?

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A knot is an embedding $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$.

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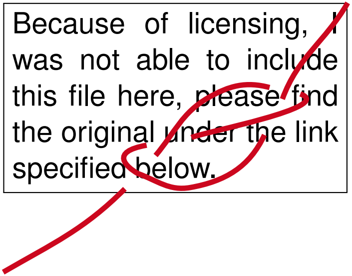

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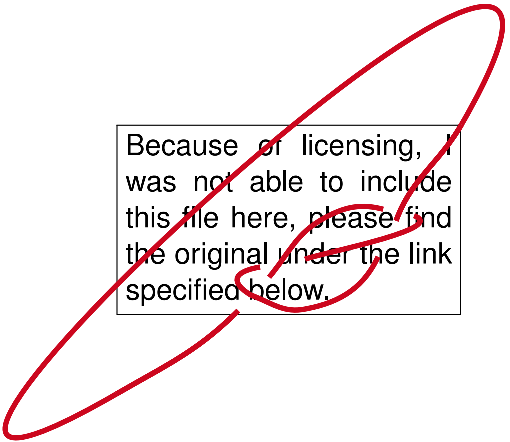

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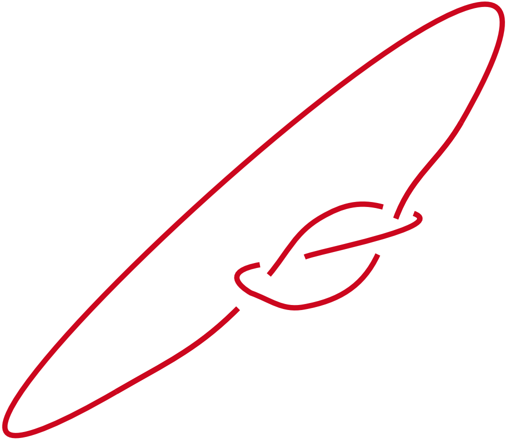

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Two knots are in the same knot class, if we can deform one into the other via ambient isotopies.

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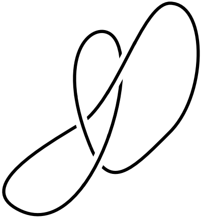

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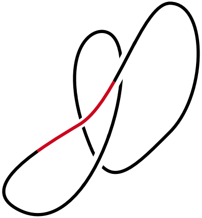

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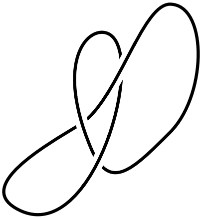

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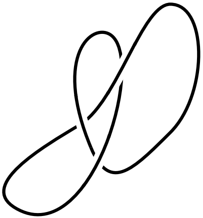

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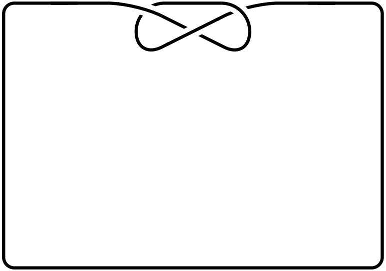

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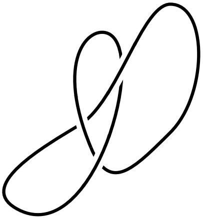

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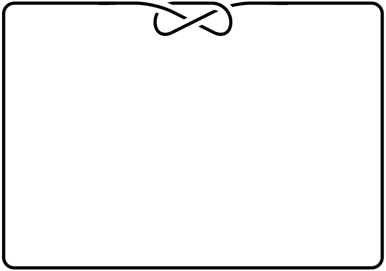

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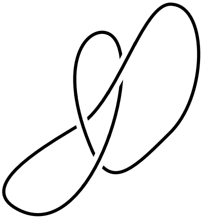

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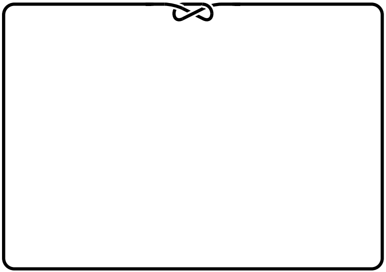

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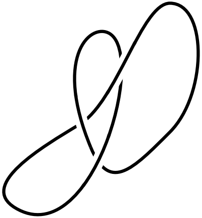

How to Tell Knots Apart?

Geometric knot theory: Knot Energies

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Image: Paweł Strzelecki, Heiko von der Mosel - Menger curvature as a knot energy, Figure 10, Physics Reports

How to Tell Knots Apart?

Geometric knot theory: Knot Energies

A functional that

- is finite on all smooth knots
- approaches infinity when the arguments uniformly approach a curve with a self-intersection
- is bounded below

is called a knot energy.

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Tight knot energies penalize pulling knots tight.

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Generalized integral Menger curvature [Blatt, Reiter 2015]

Let $p, q > 0$, $\gamma \in \mathrm{AC}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Then, the generalized integral Menger curvature is defined as

$$\mathrm{intM}^{(p,q)}(\gamma) := \iiint_{\gamma(\mathbb{R}/\mathbb{Z})^3} \frac{1}{R^{(p,q)}(x,y,z)} dH^1(x)\, dH^1(y)\, dH^1(z)$$

where

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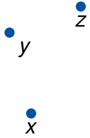

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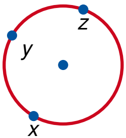

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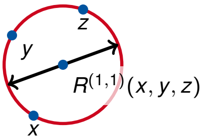

The Space Associated with $\mathrm{intM}^{(p,q)}$

For $p \in (\frac{2}{3}q + 1, q + \frac{2}{3})$, $q > 1$, $\mathrm{intM}^{(p,q)}$ is associated with $W^{\frac{3p-2}{q}-1,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, a fractional Sobolev space. [Blatt, Reiter 2015]

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$W^{k+s,q} \subseteq W^{k,q}$ is characterized by

$$\left[\gamma^{(k)}\right] := \left(\int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\left|\gamma^{(k)}(u+w) - \gamma^{(k)}(u)\right|^q}{|w|^{1+s\cdot q}} dw\, du\right)^{\frac{1}{q}} < \infty$$

$\left[\gamma^{(k)}\right]$ is called Gagliardo or Slobodeckiĭ seminorm.

Abbreviations

For $q = 2$, we have the Hilbert space $\mathcal{H} := W^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Set

- $U := W_{\mathrm{ir}}^{\frac{3}{2}p-2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, the subset of injective and regular (i.e. positive speed) curves and
- $E_p := \mathrm{intM}^{(p,2)}$.

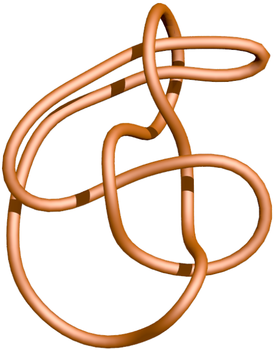
The Problem

We want a solution to

$$\begin{aligned} \frac{d}{dt}\gamma(\cdot,t) &= -\nabla E_p(\gamma(\cdot,t)), \\ \gamma(\cdot,0) &= \gamma_0 \in U. \end{aligned} \qquad \text{(Flow)}$$

Such a flow would preserve the knot class and decrease E_p as quickly as possible.

The Flow for $E_{2.6\overline{3}}$



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Additionally, we have that

$$\big(t \mapsto \gamma(\cdot, t)\big) \in C^{1,1}\left([0, \infty), \mathcal{H}\right)$$

and, for all $x \in \mathbb{R}/\mathbb{Z}$, $t \in [0, \infty)$,

$$\left|\frac{d}{dx}\gamma(x, t)\right| = \left|\frac{d}{dx}\gamma_0(x)\right|.$$

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Long Time Existence via Projected Gradients

A Quantified Picard-Lindelöf Theorem

Picard-Lindelöf Theorem

Let $A : B_r(z) \to \mathcal{H}$ be Lipschitz continuous. Then, the differential equation

$$\dot{u}(t) = A(u(t))$$
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Existence time depends only on

- upper bound M for A,
- Lipschitz Radius r:

$$T \geq \frac{r}{M}$$

[cf. Cartan 1967].

Short Time Existence

E_p is $C^{1,1}_{\text{loc}}$ with

$$r_{E_p} = r_{E_p}\left(p, \overset{\searrow}{\frac{1}{\operatorname{BiLip}(\gamma_0)}}\right), \quad M_{E_p} = M_{E_p}\left(p, \overset{\nearrow}{\frac{1}{\operatorname{BiLip}(\gamma_0)}}, \overset{\nearrow}{[\gamma_0']}\right)$$

Here, $\operatorname{BiLip}(\gamma) := \inf_{\substack{x,y\in\mathbb{R}/\mathbb{Z} \\ x\neq y}} \frac{|\gamma(x)-\gamma(y)|}{|x-y|_{\mathbb{R}/\mathbb{Z}}}$.

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$\Rightarrow$ Flow from γ_0 exists at least up to time

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Controlling Existence Times

For γ parametrized by arc-length,

$$\frac{1}{\operatorname{BiLip}(\gamma)} \leq C(p, E_p(\gamma)\nearrow)$$

and

$$[\gamma'] \leq C(p, E_p(\gamma)\nearrow),$$

[Blatt, Reiter 2015].

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Projected Gradients [Neuberger 1997]

$\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces, $\widetilde{U} \subseteq \mathcal{H}_1$ open, $E : \widetilde{U} \to \mathbb{R}$, $S : \widetilde{U} \to \mathcal{H}_2$, both Fréchet differentiable.

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Hence,

$$S(u(t)) = S(u(0))$$

for all t where $u(t)$ exists.

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Is the new RHS locally Lipschitz?
Yes, if $S \in C^{1,1}_{\text{loc}}$ and $DS(x_0)$ has a right-inverse Y.

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The Lipschitz radius around $x_0 \in \widetilde{U}$ is

$$r_{\text{RHS}} = r_{\text{RHS}}(\|DS(x_0)\|^{\searrow}, \|Y(x_0)\|^{\searrow}, L_{DS}(x_0)^{\searrow}, r_E(x_0)^{\nearrow}).$$

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Thus, the existence time at x_0 is at least

$$T(M_E\overset{\searrow}{(x_0)}, \|DS\overset{\searrow}{(x_0)}\|, \|Y\overset{\searrow}{(x_0)}\|, L_{DS}\overset{\searrow}{(x_0)}, r_E\overset{\nearrow}{(x_0)}).$$

The Logarithmic Strain Constraint

From now on, let

$$\begin{aligned} S : U &\to W^{\frac{3}{2}p-3,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}), \\ \gamma &\mapsto \left(x \mapsto \ln\left(\left|\gamma'(x)\right|\right)\right), \end{aligned}$$

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New problem:

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Thank you for your time and attention!

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